Science One Math

Jan 28 2019

Last time: $\int \tan^m(x) \sec^n(x) dx$

For any m, and $\boldsymbol{n}$ **even**

$\int \tan^m(x) \sec^n(x) dx = \int \tan^m(x) \sec^{n-2}(x) \sec^2(x) dx$

convert powers of secant into powers of tangent using

$\tan^2(x) + 1 = \sec^2(x)$ then sub $u = \tan(x)$

For any n, and $\boldsymbol{m}$ **odd**

$\int \tan^m(x) \sec^n(x) dx = \int \tan^{m-1}(x) \sec^{n-1}(x) \sec(x) \tan(x)\, dx =$

convert powers of tangent into powers of secant by using

$\tan^2(x) = \sec^2(x) - 1$

What if $\boldsymbol{n}$ **odd** and $\boldsymbol{m}$ **even?** Use integration by parts!

If m is even, n is odd $\int \tan^m(x)\sec^n(x)dx$ use IBP and reduction formula

Example:

$\int \tan^2(x)\sec^3(x)dx = \int(\sec^2(x) - 1)\sec^3(x)dx = \int \sec^5(x) - \sec^3(x)\, dx$

$\int \sec^3(x)dx = \int \sec(x)\sec^2(x)dx$ $\quad$ $u = \sec x$ $\quad$ dv = $\sec^2(x)dx$

du = $\tan(x)\sec(x)\,dx$ $\quad$ v = $\tan(x)$

= $\sec(x)\tan(x) - \int \tan(x)\tan(x)\sec(x)\,dx$ =

= $\sec(x)\tan(x) - \int \tan^2(x)\sec(x)\,dx$ $\quad$ [$\tan^2(x) = \sec^2(x) - 1$]

= $\sec(x)\tan(x) - \int \sec^3(x)dx + \int \sec(x)\,dx$

So $\int \sec^3(x)dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx$

it reduces to computing $\int \sec(x)dx$ …we need some trickery here… $u = \sec x + \tan x$,

$du = (\sec x\ \tan x + \sec^2 x)dx$

$$\int \sec x\, dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x} dx = \int \frac{sec^2 x + \sec x \tan x}{\sec x + \tan x} dx = \int \frac{du}{u} = \ln|u| + C = \cdots$$

Last time: Trigonometric integrals

Integrals of $\sin^n(x)\cos^m(x)$ or $\tan^n(x)\sec^m(x)$ are computable!

Strategies:

- u-substitution
- trig identities
- (sometimes) IBP

Last time: Trigonometric integrals

Not all integrals with trigonometric functions are computable, e.g.

$$\int \sin(x^2)\,dx\,, \qquad \int \cos(x^2)\,dx\,, \qquad \int \frac{\sin(x)}{x}\,dx$$

We can only approximate these integrals numerically.

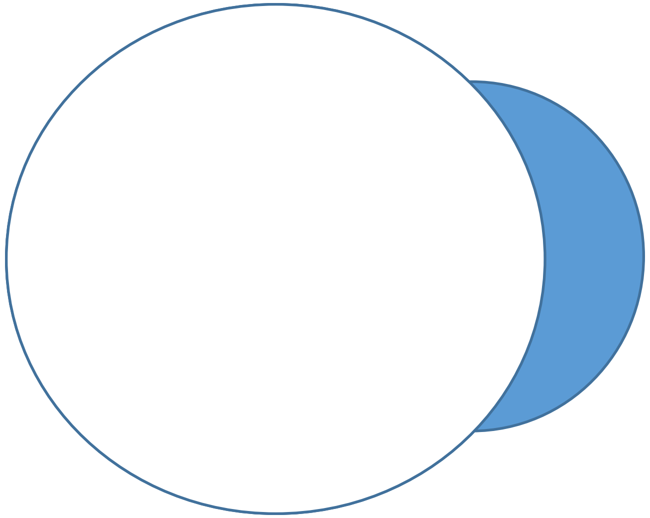
When do we see trigonometric integrals?

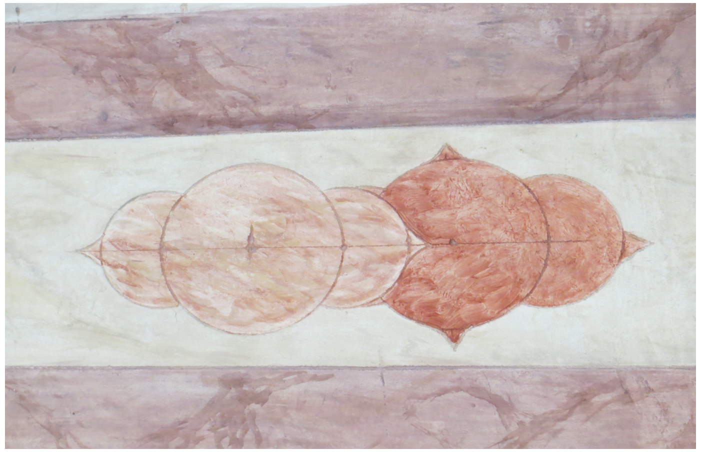
Many computations lead to trigonometric integrals:

- Analysis of oscillating systems (waves, alternating current circuits, etc.)
- Fourier analysis (widely used technique where a periodic function is decomposed in terms of infinite sums of powers of sines and cosines)
- Integrals with roots

Can we compute area of this shape?

The shaded area is called a *"lune"*





The Ducal Palace – Mantua, Italy

Russ Adams in Pike County, IL

It reduces to $\int \sqrt{1 - x^2}dx$

need a new strategy! Trigonometric substitutions

Science One Math

Jan 30 2019

$\int_1^4 \frac{1}{x^{1/2}+x^{3/2}}\,dx$ can be written as

A) $\int_1^4 x^{-1/2} + x^{-3/2}\,dx$

B) $\int_1^4 \frac{1}{x^{1/2}} + \frac{1}{x^{3/2}}\,dx$

C) Either one—they are equivalent

D) **A and B are equivalent but they are not a correct simplification of** $\boldsymbol{\frac{1}{x^{1/2}+x^{3/2}}}$

Today’s goal: Trigonometric substitutions

…how to compute integrals with roots.

How to we compute integrals with roots?

$\int \sqrt{1-x^2}\,dx$

$\int x\sqrt{1-x^2}\,dx$ ⟸ start with this one

$\int x^2\sqrt{1-x^2}\,dx$

$\int x^3\sqrt{1-x^2}\,dx$

$$\int x\sqrt{1-x^2}\,dx = \int -\tfrac{1}{2}\sqrt{u}\,du = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(1-x^2)^{3/2} + C$$

$u = 1 - x^2,\ du = -2x\,dx$

Does substitution work for the other integrals?

$$\int \sqrt{1-x^2}\,dx$$

$$\int x^2\sqrt{1-x^2}\,dx$$

$$\int x^3\sqrt{1-x^2}\,dx$$

$\int x\sqrt{1-x^2}\,dx = \int -\frac{1}{2}\sqrt{u}\,du = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(1-x^2)^{3/2} + C$

$u = 1-x^2,\ du = -2x\,dx$

Does substitution work for the other integrals?

$\int \sqrt{1-x^2}\,dx$ no!

$\int x^2\sqrt{1-x^2}\,dx$ no!

$\int x^3\sqrt{1-x^2}\,dx$ yes! $\int x^2\sqrt{1-x^2}\,x\,dx = \int -\frac{1}{2}(1-u)\sqrt{u}\,du =\ldots$

When u-sub doesn't work, we need a **different type of substitution…**

An observation:

$$\int x^3\sqrt{1-x^2}\,dx = \int x^2\,\sqrt{1-x^2}\,x\,dx$$

$$1 - x^2 = u^2\ ,\ x^2 = 1 - u^2$$

$$-2\,x\,dx \;=\; 2\,u\,du$$

$$= \int -\tfrac{1}{2}(1-u^2)\sqrt{u^2}\,u\,du \;= \int -\tfrac{1}{2}(1-u^2)\,u^2\,du = \ldots$$

So maybe to compute the other integrals we could use a substitution of the form

$$1 - x^2 = (\ldots)^2$$

An observation:

$$\int x^3\sqrt{1-x^2}\,dx = \int x^2\,\sqrt{1-x^2}\,x\,dx$$

$$1-x^2 = u^2\ ,\ x^2 = 1-u^2$$

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$$= \int -\tfrac{1}{2}(1-u^2)\sqrt{u^2}\,u\,du \ = \int -\tfrac{1}{2}(1-u^2)\,u^2\ du = \ldots$$

So maybe to compute the other integrals we need to use a substitution of the form

$$1-x^2 = (\ldots)^2$$

$$1-\ \sin^2(\theta) = \ \cos^2(\theta)$$

Trigonometric substitutions

$x = \sin\theta$, $\mathrm{d}x = \cos\theta\, d\theta$ then $1 - x^2 = 1 - \sin^2(x) = \cos^2(x)$

$\int \sqrt{1 - x^2}\, dx$ = $\int \sqrt{1 - sin^2(\theta)}\, \cos\theta\, d\theta$ =

$= \int \sqrt{cos^2(\theta)} \cos\theta\, d\theta$ = $\int cos^2(\theta) d\theta$

Note $\sqrt{cos^2(\theta)}$ = $|\cos\theta|$. We take $|\cos\theta|$ = $\cos\theta$, i.e. $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

…and now we compute the trigonometric integral….

"Best" strategy to compute $\int cos^2(\theta)d\theta$ is…

a) By parts
b) By using trig identity $\cos^2(x) = 1 - \sin^2(x)$
c) By using double-angle identity and then substitution
d) By substitution right away
e) By parts and then reduction

“Best” strategy to compute $\int \cos^2(\theta) d\theta$ is…

a) By parts
b) By using trig identity $\cos^2(x) = 1 - \sin^2(x)$
c) By using double-angle identify and then substitution
d) By substitution right away
e) By parts, and then reduction

$$\int \sqrt{1-x^2}\,dx = \int \cos^2(\theta)d\theta = \int \frac{1}{2}\,[1+\cos(2\theta)]\,d\theta =$$

$$= \frac{1}{2}\,[\theta + \frac{1}{2}\sin(2\theta)] + C$$ ...convert back to the original variable x...

Note $x = \sin\theta$, hence

$$\theta = \arcsin(x)$$

$$\cos\theta = \sqrt{1-\sin^2(\theta)} = \sqrt{1-x^2}$$

Therefore, recalling $\sin(2\theta) = 2\sin(\theta)\cos\theta$,

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\theta + \frac{1}{4}2\sin(\theta)\cos\theta + C = \frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2} + C.$$

$$\int x^2\sqrt{1-x^2}\,dx$$

$$\int \sqrt{9-x^2}\,dx$$

$$\int x^2\sqrt{1-x^2}\,dx$$

$$x = \sin\theta\,,\ \mathrm{d}x = \cos\theta\ d\theta$$

$$=\int \sin^2(\theta)\sqrt{1-\sin^2(\theta)}\ \cos\theta\ d\theta = \int \sin^2(\theta)\sqrt{\cos^2(\theta)}\cos\theta\ d\theta =$$

$$= \int \sin^2(\theta)\cos^2(\theta)d\theta = \int(1-\cos^2(\theta))\cos^2(\theta)d\theta =$$

$$= \int \cos^{\mathbf{2}}(\theta)d\theta - \int \cos^{\mathbf{4}}(\theta)d\theta = [\text{from last week}]$$

$$=\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) - \frac{1}{32}(12\theta + 8\sin(2\theta) + \sin(4\theta)) + C =$$

$$=\frac{1}{8}\ \theta - \frac{1}{32}\sin(4\theta) + C =$$

$$= \frac{1}{8}\ \theta - \frac{1}{32}2\sin(2\theta)\cos(2\theta) + C\ =$$

$$= \frac{1}{8}\ \theta - \frac{1}{16}2\sin(\theta)\cos(\theta)\ (1-2\sin^2(\theta)) + C\ =$$

$$= \frac{1}{8}\ \arcsin(x) - \frac{1}{8}x\sqrt{1-x^2}\ (1-2x^2) + C$$

What substitution would work for $\int \sqrt{9 - x^2}\, dx$?

a) $x = \sin(9\theta)$
b) $x = 9\sin\theta$
c) $3x = \sin\theta$
d) $x = 3\sin\theta$
e) None of the above

What substitution would work for $\int \sqrt{9 - x^2}\, dx$?

a) $x = \sin(9\theta)$
b) $x = 9\sin\theta$
c) $3x = \sin\theta$
d) $\boldsymbol{x = 3sin\,\theta}$
e) None of the above

Using the substitution $x = 3\sin\theta$, the integral $\int \sqrt{9 - x^2}\, dx$ is equivalent to...

a) $\int \sqrt{9 - 9sin^2(x)}\, dx$

b) $\int 3\sqrt{1 - sin^2(\theta)}\, d\theta$

c) $\int 3cos^2(\theta)\, d\theta$

d) $\int 9cos^2(\theta)\, d\theta$

e) $\int 9cos^3(\theta)\, d\theta$

Using the substitution $x = 3\sin\theta$, the integral $\int \sqrt{9 - x^2}\, dx$ is equivalent to...

a) $\int \sqrt{9 - 9sin^2(x)}\, dx$

b) $\int 3\sqrt{1 - sin^2(\theta)}\, d\theta$

c) $\int 3cos^2(\theta)\, d\theta$

d) $\boldsymbol{\int 9cos^2(\theta)\, d\theta}$

e) $\int 9cos^3(\theta)\, d\theta$

What substitution would simplify $\int \sqrt{9+x^2}\,dx$?

a) $x = 3\sin\theta$
b) $x = 3\cos\theta$
c) $x = 3\tan\theta$
d) $x = 3\sec\theta$
e) None of the above

What substitution would simplify $\int \sqrt{9 + x^2}\, dx$?

a) $x = 3\sin\theta$
b) $x = 3\cos\theta$
c) $\boldsymbol{x = 3\tan\theta}$
d) $x = 3\sec\theta$
e) None of the above

General case: 3 basic substitutions

- $\sqrt{a^2 - x^2} = a\sqrt{1 - \sin^2\theta}$ with $x = a\sin\theta$, where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
- $\sqrt{a^2 + x^2} = a\sqrt{1 + \tan^2\theta}$ with $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
- $\sqrt{x^2 - a^2} = a\sqrt{sec^2\theta - 1}$ with $x = a\,sec\,\theta$, where $0 \leq \theta < \frac{\pi}{2}$

$$\int \frac{dx}{(16-x^2)^{3/2}}$$

$$\int \frac{dy}{y\sqrt{1+(\ln y)^2}}$$

$$\int \frac{dx}{(16-x^2)^{3/2}} = \int \frac{dx}{16^{3/2}[1-\left(\frac{x}{4}\right)^2]^{3/2}}$$

$$x = 4\sin\theta,\ dx = 4\cos\theta\ d\theta$$

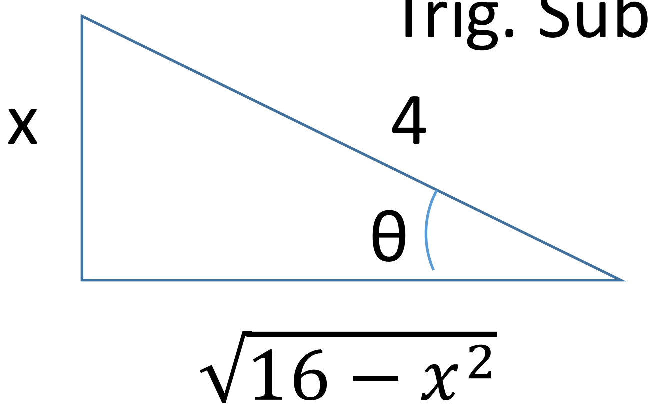
$$= \frac{1}{64}\int \frac{4\cos\theta\, d\theta}{[1-(\sin\theta)^2]^{\frac{3}{2}}} = \frac{1}{64}\int \frac{4\cos\theta\, d\theta}{\cos^3\theta} = \frac{1}{16}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{16}\tan\theta + C$$

$$\int \frac{dx}{(16-x^2)^{3/2}} = \int \frac{dx}{16^{3/2}(1-(x/4)^2)^{3/2}}$$

$$x = 4\sin\theta,\ dx = 4\cos\theta\ d\theta$$

$$= \frac{1}{64}\int \frac{4\cos\theta\, d\theta}{(1-(\sin\theta)^2)^{\frac{3}{2}}} = \frac{1}{64}\int \frac{4\cos\theta\, d\theta}{\cos^3\theta} = \frac{1}{16}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{16}\tan\theta + C$$

Trig. Substitutions are a manifestation of Pythagora's thrm.



$$x = 4\sin\theta$$

$$\tan\theta = \frac{x}{\sqrt{16-x^2}}$$

Thus

$$\int \frac{dx}{(16-x^2)^{3/2}} = \frac{1}{16}\tan\theta + C = \frac{x}{16\sqrt{16-x^2}} + C$$

$$\int \frac{dy}{y\sqrt{1+(\ln y)^2}} \qquad u = \ln y,\ du = \frac{1}{y}dy$$

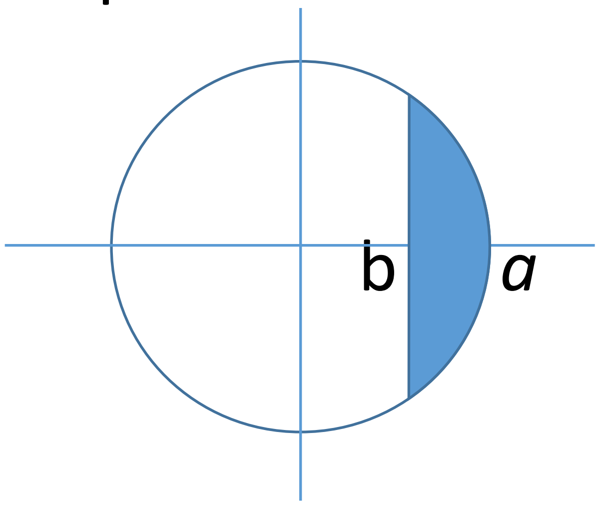
$$\int \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int \frac{du}{\sqrt{1+u^2}}$$

$$u = \tan\theta,\ du = \sec^2(\theta)d\theta,\ \ \sqrt{1+u^2} = \sqrt{1+\tan^2(\theta)} = \sqrt{\sec^2(\theta)}$$

$$\int \frac{du}{\sqrt{1+u^2}} = \int \frac{\sec^2(\theta)d\theta}{\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C =$$

$$= \ln\left|\sqrt{1+u^2} + u\right| + C = \ln\left|\sqrt{1+(\ln y)^2} + \ln y\right| + C$$

Compute the area of a circular segment (*shaded region*)



$$A = 2\int_b^a \sqrt{a^2 - x^2}\, dx = 2a\int_b^a \sqrt{1 - \left(\frac{x}{a}\right)^2}\, dx$$

$$x = a\sin\theta$$

$$2a^2 \int_{x=b}^{x=a} cos^2\theta\, d\theta = a^2\left(\theta + \tfrac{1}{2}\sin(2\theta)\right)\Big|_{x=b}^{x=a}$$

$$= a^2\left[\arcsin(x/a) + \frac{x}{a}\sqrt{1-(x/a)^2}\right]_{x=b}^{x=a}$$

$$= a^2\left(\frac{\pi}{2} + 0 - \arcsin\left(\frac{b}{a}\right) - \frac{b}{a}\sqrt{1-\left(\frac{b}{a}\right)^2}\right) =$$

$$\frac{\pi}{2}a^2 - a^2\arcsin\left(\frac{b}{a}\right) - b\sqrt{a^2 - b^2}$$