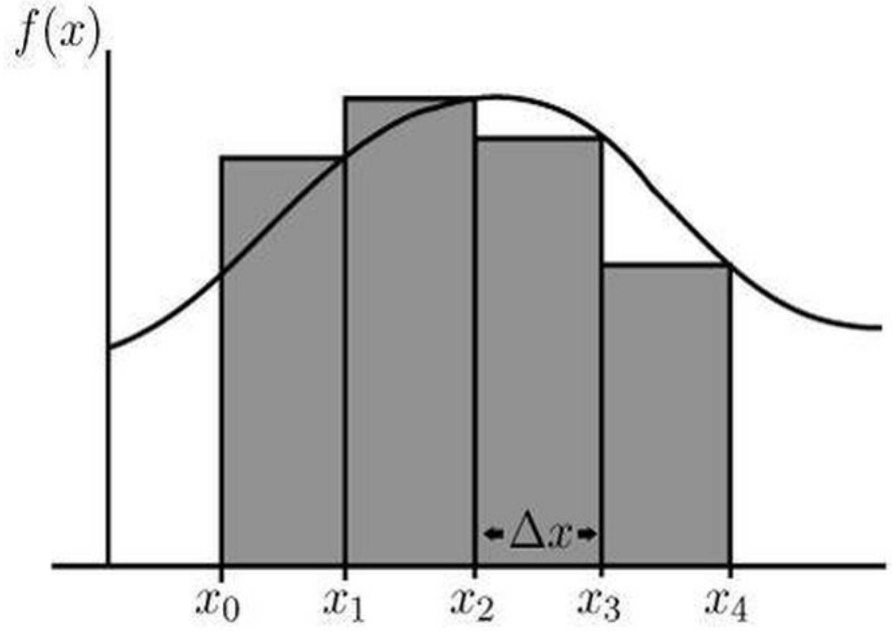
Science One
Integral Calculus

January 7, 2019

Last time: approximating area

Key ideas:

- Divide region into n vertical strips
- Approximate each strip by a rectangle
- Sum area of rectangles
- Increase n (take limit for $n \to \infty$)



more formally….

Consider the region under the curve $y = f(x)$ above $[a, b]$.

- Take n vertical strips of equal width $\Delta x = (b-a)/n$
- Consider n intervals: $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots [x_{i-1}, x_i], \ldots [x_{n-1}, x_n]$.
- Build n rectangles and sum all areas

$S_n = \Delta x\, f(x_1^*) + \Delta x\, f(x_2^*) + \ldots + \Delta x\, f(x_i^*) + \ldots + \Delta x\, f(x_n^*) = \sum_{i=1}^{n} f(x_i^*)\, \Delta x$

where sample point x_i^* is *any* number in the interval $[x_{i-1}, x_i]$.

Defn: The **area** S of the region under the graph of f above $[a, b]$ is

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\, \Delta x = \int_a^b f(x)\,dx$$

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Riemann Sum

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Definite Integral

The Riemann sum $\sum_{i=1}^{n} f(x_i^*)\ \Delta x$

sample point x_i* is *any* number in the interval $[x_{i-1}, x_i]$

The Riemann sum ➟ a method for computing an **approximation** of area (approximation of a definite integral)

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The Riemann sum ➟ a method for computing an **approximation** of area

Specific types of Riemann Sums

-) if $x_i^* = x_i = a + i\Delta x$ ➟ "right Riemann sum"

  (height of rectangle = right edge of the strip)

-) if $x_i^* = x_{i-1} = a + (i-1)\Delta x$ ➟ "left Riemann sum"

  (height of rectangle = left edge of the strip)

-) if $x_i^* = (x_i + x_{i-1})/2$ ➟ "midpoint sum"

Which of the following expressions represents the area below the curve of $\sin(5x)$ on $[0, \pi/5]$?

a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin\left(\pi + \frac{5\pi\, i}{n}\right)$

b) $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{\pi\, i}{n}\right)$

c) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{5n} \sin\left(\frac{5\pi\, i}{n}\right)$

d) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{5n} \sin\left(\frac{\pi\, i}{n}\right)$

e) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin\left(\frac{5\pi\, i}{n}\right)$

The definite integral $\int_a^b f(x)dx$: Terminology

Defn: The **area** S of the region under the graph of f above $[a\,,b]$ is

$$S = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\ \Delta x\ = \int_a^b f(x)dx$$

Definite Integral

$$\int_a^b f(x)dx$$

integrand

The definite integral $\int_a^b f(x)dx$: Terminology

Defn: The **area** S of the region under the graph of *f* above $[a\,,b]$ is

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Definite Integral

$$\int_a^b f(x)dx$$

limits of integration

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Definite Integral

$$\int_a^b f(x)dx$$

(dummy) integration variable

Which of the following correctly expresses the limit

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} \cos\left(1 + \frac{i}{n}\right)$ as a definite integral?

a) $\int_0^2 \cos(x)\, dx$
b) $\int_0^2 \cos(x+1)\, dx$
c) $\int_1^3 \cos(x)\, dx$
d) $\int_1^2 \cos(x)\, dx$
e) $\int_1^2 2\cos(x)\, dx$

Can you think of another integral that can be expressed as the limit above?

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Definite Integral

The above definition is for $f(x) > 0$ for all x in $[a\,, b]$.

What if $f(x) < 0$ for some x in $[a\,, b]$?

What if $f\ (x) < 0$ on $[a\ , b]$?

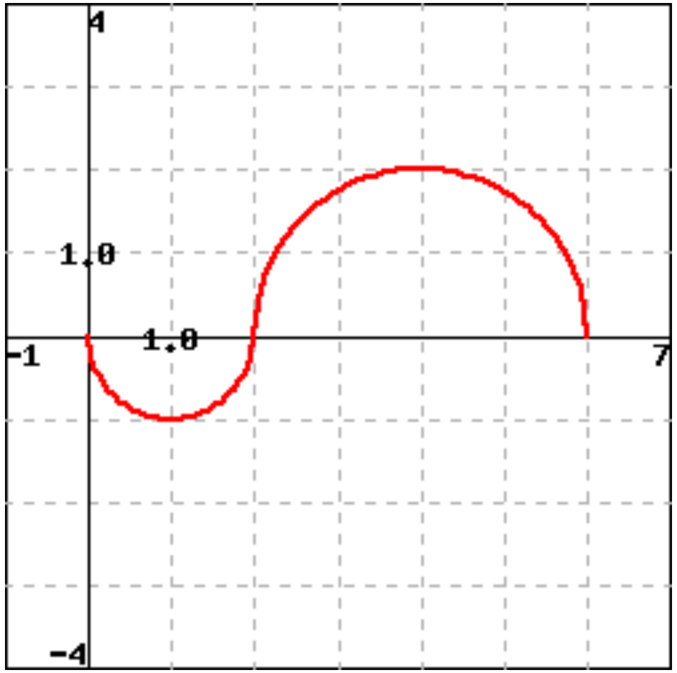
If $f\ (x) < 0$ on $[a\ , b]$ then curve is below x-axis...

➠ the height of the approximating rectangles is *negative*...

➠ the Riemann sum is the sum of the *negatives* of the areas of the rectangles that lie below the x-axis (and above the curve)

$\int_a^b f(x)dx < 0$ "signed area"

$\left|\int_a^b f(x)dx\right| =$ area of region below x-axis, above curve $y = f(x)$ on $[a, b]$



Using the definition of integral as area, evaluate

$\int_0^2 f(x)dx =$

a) π

b) $\pi/2$

c) *Not defined*

d) $-\pi$

e) $-\pi/2$

When does the limit of Riemann sums exist?

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x \;=\int_a^b f(x)dx$$

Thrm: If $f(x)$ is defined on $[a,b]$ and

- $f(x)$ is continuous on $[a,b]$, or
- $f(x)$ has a finite number of jump discontinuities on $[a,b]$

then the limit of Riemann sums exists and we say

$f(x)$ is **integrable** on $[a,b]$.

Evaulate $\int_1^4 \mid x - 3 \mid dx$

Evaulate $\int_1^4 |\ x - 3\ |\ dx$ = $\int_1^3 (3 - x)\ dx + \int_3^4 (x - 3)\ dx$ =

[using areas] $= \frac{1}{2}(2)(2) + \frac{1}{2}(1)(1) = \frac{5}{2}$

Some properties of the integral:

1) $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

2) $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

3) If $f(x) = c = constant$, then $\int_a^b c\, dx = c(b-a)$ ➡ $\int_a^b 1 \cdot dx = b - a$

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➡ $\int_a^b [Af(x) + Bg(x)]\, dx = \mathrm{A} \int_a^b f(x)dx + \mathrm{B} \int_a^b g(x)dx$

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1) $\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

2) $\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

3) If $f(x) = c = constant$, then $\int_a^b c\,dx = c(b-a)$

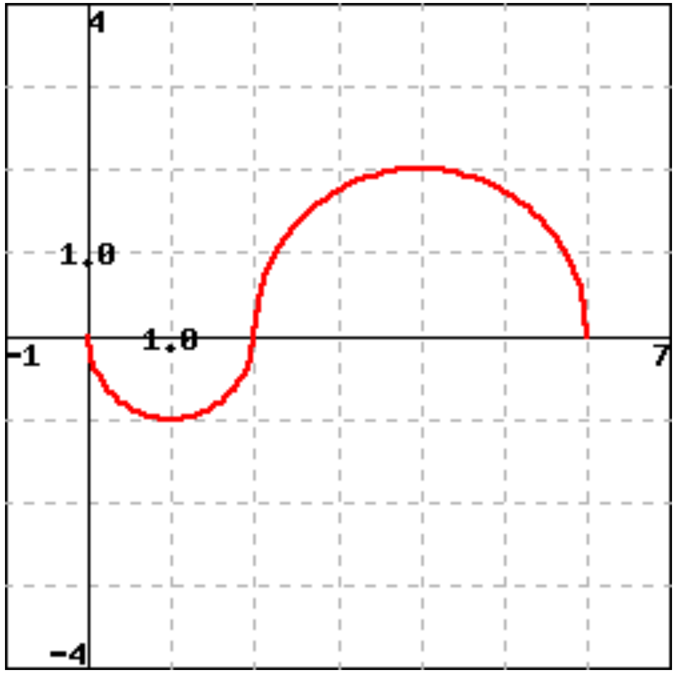
➡ $\int_a^b Af(x)+Bg(x)\,dx = \mathrm{A}\int_a^b f(x)dx + \mathrm{B}\int_a^b g(x)dx$

4) $\int_b^a f(x)dx = -\int_a^b f(x)dx$

5) $\int_a^a f(x)dx = 0$

6) $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

Evaluating integrals using known areas



1) Evaluate $\int_1^4 f(x)dx$

a) 4π
b) $3\pi/4$
c) $5\pi/4$
d) 2π

2) Evaluate $\int_{-1}^{1} 3 + \sqrt{1 - x^2}dx$

Evaluate $\int_{-1}^{1} 3 + \sqrt{1 - x^2}\,dx$ *(no need to find antiderivatives here)*

$$\int_{-1}^{1} 3 + \sqrt{1 - x^2}dx = \int_{-1}^{1} 3dx + \int_{-1}^{1} \sqrt{1 - x^2}\, dx$$

$$\int_{-1}^{1} 3dx = \textit{area rectangle of base 2 and height 3} = 2x3 = 6$$

$$\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \textit{area half circle of radius 1 centred at origin} = \frac{\pi}{2}$$

$$\int_{-1}^{1} 3 + \sqrt{1 - x^2}dx = 6 + \pi/2$$

Comparison properties of the integral

1. If $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)dx \geq 0$.
2. If $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
3. If $m \leq f(x) \leq M$ on $[a, b]$, then

$$m(b - a) \leq \int_a^b f(x)dx \leq M\,(b - a).$$

…useful to bound integrals.

Estimate $\int_0^1 e^{-x^2} dx$.

Observe e^{-x^2} is a decreasing function on [0, 1], that is

$$e^{-1} \leq e^{-x^2} \leq 1$$

Thus

$$e^{-1}(1-0) \leq \int_0^1 e^{-x^2} dx \ \leq 1\ (1-0).$$

We estimate

$$0.3679 \leq \int_0^1 e^{-x^2} dx \leq 1$$

Recap: What have we learned so far?

- The definite integral is defined as a limit of Riemann sums
- Riemann sums can be constructed using any point in a subinterval
- Riemann sums provide a method to approximate an integral
- Any (piece-wise) continuous function is integrable
- $\int_a^b f(x)dx$ represents a "signed area" of the region

Sad news: Evaluating the limit of Riemann sums is hard!

Good news: there is an easier way to compute integrals…

The Fundamental Theorem of Calculus

Theorem

Let f be continuous on an interval $\mathcal{I}$ containing *a*.

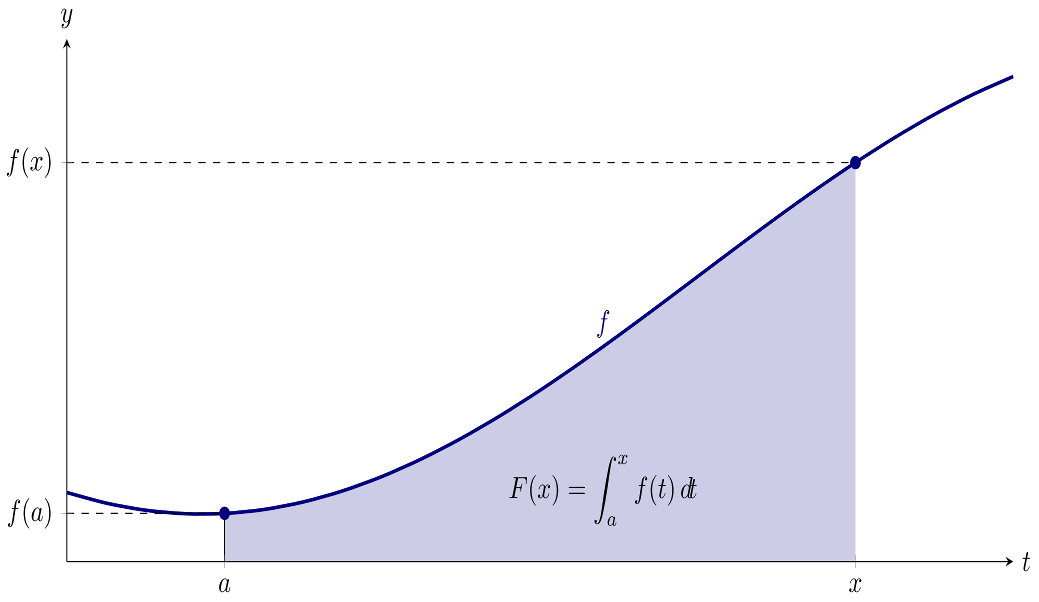
1. Define $F(x) = \int_a^x f(t)dt$ on $\mathcal{I}$. Then F is differentiable on $\mathcal{I}$ with

   $F'(x) = f(x)$.

2. Let G be any antiderivative of f on $\mathcal{I}$. Then for any *b* in $\mathcal{I}$

$$\int_a^b f(t)dt = G(b) - G(a)$$

FTC part I: The area function



if $f(x)$ is continuous on $\mathcal{I}$

let $F(x)$ = area under f(x) on [a, x]

$$F(x) = \int_a^x f(t)dt$$

$F(x)$ is called "accumulation function"

In science there are many functions defined as an integral:

Error function $Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ (probability and statistics)

Sine integral function $Si(x) = \int_0^x \frac{\sin t}{t} dt$ (signal processing)

Fresnel functions $S(x) = \int_0^x \sin(t^2)\, dt$

$C(x) = \int_0^x \cos(t^2)\, dt$ (theory of diffraction)

Natural logarithm $ln(x) = \int_1^x \frac{1}{t} dt$ for $x > 0$

FTC part I: The derivative of the area function

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Proof

Suppose f is continuous (and positive) on an interval containing a.

What is the area below the curve $y = f(t)$ on $[a, x]$?

$$\text{Area} = F(x) = \int_a^x f(t)dt\ .$$

At what rate is the area changing with respect to x? $F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$

Key Observation:

$F(x+h) - F(x)$ is a difference of areas approximated by a rectangle of area $h \cdot f(x)$.

Hence, in the limit $h \to 0$, we get $F'(x) = f(x)$.

Problem: Let $F(x) = \int_{-1}^{x} t^3 dt$. Considering the interval [-1, 1], find where F is increasing? decreasing? Where does F have local extrema?

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Recall: (FTC part I) If $F(x) = \int_{a}^{x} f(t)dt$, F is differentiable on $[a, x]$ with $F'(x) = f(x)$.

Problem: Let *F(x)* = $\int_{-1}^{x} t^3 dt$. Considering the interval [-1, 1], find where F is increasing? decreasing? Where does F have local extrema?

Recall: (FTC part I) If $F(x) = \int_{a}^{x} f(t)dt$, F is differentiable on $[a, x]$ with $F'(x) = f(x)$.

Solution:

F is increasing when $F' > 0$. By FTC, $F'(x) = x^3$.

$x^3 > 0$ when $x > 0 \Rightarrow F$ is increasing on (0,1), decreasing on (-1,0).

$F(0)$ is a local minimum.

$F(0) = \int_{-1}^{0} t^3 dt$ = ?

We need to compute the definitive integral. Let's use the second part of FTC.

FTC part II: $\int_a^b f(t)dt = G(b) - G(a)$, where G'(x)=f(x).

Proof:

If $F(x) = \int_a^x f(t)dt$, we know $F'(x) = f(x)$. That is, *F* is an antiderivative of *f*.

Suppose *G* is *any* antiderivative of *f* on I. Then $G(x) = F(x) + C$ for some constant *C*.

If $x = a$, $F(a) = 0$ so $G(a) = C$.

If $x = b$, $F(b) = \int_a^b f(t)dt \Rightarrow G(b) = F(b) + G(a) = \int_a^b f(t)dt + G(a)$.

$$\Rightarrow \int_a^b f(t)dt = G(b) - G(a).$$

…so now we know how to compute definite integrals without Riemann sums!

Previous problem:

Let $F(x) = \int_{-1}^{x} t^3 \, dt$. Compute F(0).

$$F(0) = \int_{-1}^{0} t^3 dt = \frac{1}{4} t^4 \, \Big|_{-1}^{0} = 0 - \frac{(-1)^4}{4} = -\frac{1}{4}$$

Recap:
the derivative undoes the integral, and vice versa!

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Evaluating definite integrals involves finding antiderivatives (indefinite integrals)…instead of “integration”, we should call it “antidifferentiation”!