Science One
Integral Calculus

January 9, 2019

Recap: What have we learned so far?

- The definite integral is defined as a limit of Riemann sums
- Riemann sums can be constructed using any point in a subinterval
- Riemann sums provide a method to approximate an integral
- Any (piece-wise) continuous function is integrable
- $\int_a^b f(x)dx$ represents a "signed area" of the region

Sad news: Evaluating the limit of Riemann sums is hard!

Good news: there is an easier way to compute integrals…

The Fundamental Theorem of Calculus

Theorem

Let f be continuous on an interval $\mathcal{I}$ containing *a*.

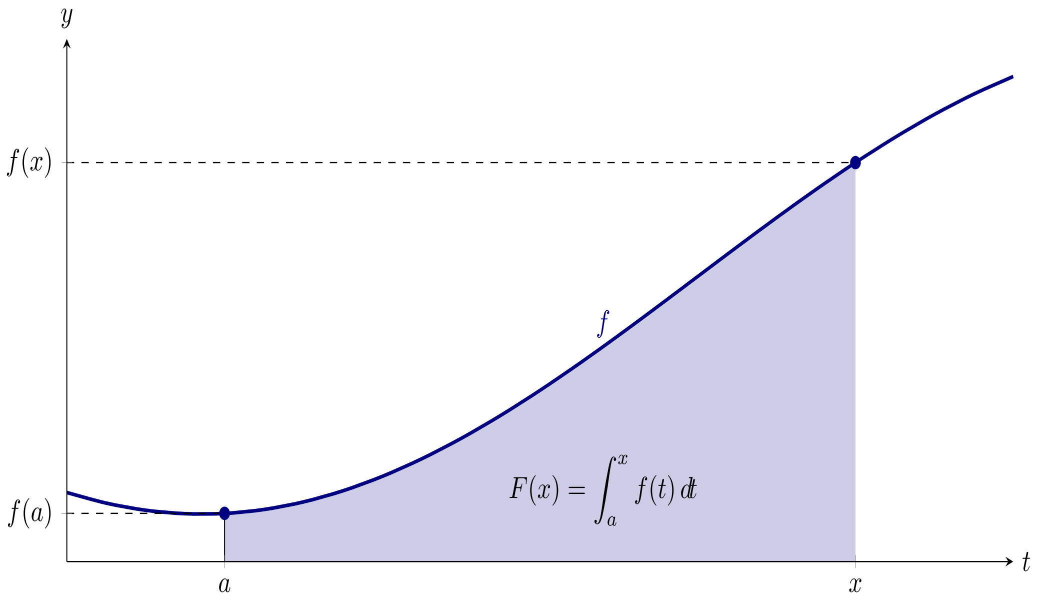
1. Define $F(x) = \int_a^x f(t)dt$ on $\mathcal{I}$. Then F is differentiable on $\mathcal{I}$ with

   $F'(x) = f(x)$.

2. Let G be any antiderivative of f on $\mathcal{I}$. Then for any *b* in $\mathcal{I}$

$$\int_a^b f(t)dt = G(b) - G(a)$$

FTC part I: The area function



if $f(x)$ is continuous on $\mathcal{I}$

let $F(x)$ = area under $f(x)$ on $[a, x]$

$$F(x) = \int_a^x f(t)dt$$

$F(x)$ is called "accumulation function"

In science there are many functions defined as an integral:

Error function $Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ (probability and statistics)

Sine integral function $Si(x) = \int_0^x \frac{\sin t}{t} dt$ (signal processing)

Fresnel functions $S(x) = \int_0^x \sin(t^2)\, dt$

$C(x) = \int_0^x \cos(t^2)\, dt$ (theory of diffraction)

Natural logarithm $ln(x) = \int_1^x \frac{1}{t} dt$ for $x > 0$

FTC part I: The derivative of the area function

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Proof

Suppose f is continuous (and positive) on an interval containing a.

What is the area below the curve $y = f(t)$ on $[a, x]$? Area $= F(x) = \int_a^x f(t)dt$.

At what rate is the area changing with respect to x? $F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$

Key Observation:

$F(x+h) - F(x)$ is a difference of areas approximated by a rectangle of area $h \cdot f(x)$.

Hence, in the limit $h \to 0$, we get

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} \frac{hf(x)}{h} = f(x) .$$

An example…

Using known areas we know that $\int_1^2 f(t)dt = 1.5$ when $f(t) = t$.

Now consider $\int_1^z t\, dt$. Using known areas we compute

$\int_1^z t\, dt = \frac{1}{2}(1+z)(z-1) = \frac{1}{2}(z^2-1) = F(z)$.

Using appropriate differentiation rules we find

$$\frac{dF}{dz} = z$$

That is $\frac{dF}{dz} = f(z)$.

FTC part II: $\int_a^b f(t)dt = G(b) - G(a)$ if $G'(x) = f(x)$.

Proof:

If $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$. That is, F is an antiderivative of f.

Suppose G is *any* antiderivative of f on I. Then $G(x) = F(x) + C$ for some C.

If $x = a$, $F(a) = 0$ so $G(a) = C$.

If $x = b$, $F(b) = \int_a^b f(t)dt \Rightarrow G(b) = F(b) + G(a) = \int_a^b f(t)dt + G(a)$.

$$\Rightarrow \int_a^b f(t)dt = G(b) - G(a).$$

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…so now we know how to compute definite integrals without using Riemann sums!

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Recall: (FTC part I)

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Recall: (FTC part I) If $F(x) = \int_{a}^{x} f(t)dt$, F is differentiable on $[a, x]$ with $F'(x) = f(x)$.

Solution: *F* is increasing when $F' > 0$. By FTC, $F'(x) = x^3$.

$x^3 > 0$ when $x > 0 \Rightarrow F$ is increasing on (0,1), decreasing on (-1,0).

$F(0)$ is a local minimum.

$F(0) = \int_{-1}^{0} t^3 dt$ = ?

We need to compute the definitive integral. Let's use the second part of FTC.

Let $F(x) = \int_{-1}^{x} t^3 dt$. Compute $F(0)$.

Observe $F(0) = \int_{-1}^{0} t^3 dt$.

An antiderivative of t^3 is $\frac{1}{4}t^4$. Then by FTC (Part II) we have

$$F(0) = \int_{-1}^{0} t^3 dt = \frac{1}{4}t^4 \Big|_{-1}^{0} = 0 - \frac{(-1)^4}{4} = -\frac{1}{4}$$

Recap:
the derivative undoes the integral, and vice versa!

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Evaluating definite integrals involves finding antiderivatives (indefinite integrals)…instead of “integration”, we should call it “antidifferentiation”!

definite vs indefinite integral

The **definite integral**: $\int_a^b f(x)dx$ **This is a number!**

The **indefinite integral**: $\int f(x)dx$ **This is a family of functions**!

Problems:

1) Find the area of the region under y = 3x – x^2 and above x-axis. [Ans: 27/6]
2) Find the area of the region under y= $\frac{5}{x^2+1}$ and above y = 1. [Ans: 10arctan(2)-4]
3) Find $\frac{d}{dx} x^2 \int_{-4}^{5x} e^{-t^2}$dt
4) Find $\frac{d}{dx} \int_{x}^{x^3} e^{-t^2}$dt
5) If $\int_a^b f(t)dt$=0 and f is continuous on [a, b], prove there is a point c in [a,b] with $f(c) = 0$.

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(chemistry) Let [C](t) be the concentration of the product of a chemical reaction at time t, then

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(biology) Let p(t) be a population size at time t, then

$\int_{t_1}^{t_2} \frac{dp}{dt} dt = p(t_2) - p(t_1)$ is net change in population size.

some more applications…

- **Areas between curves**
- **Volumes of solids**
- **Work** done by non constant force
- Average value of a function
- Arc length
- Surface area of solids
- Hydrostatic pressure
- **Probability density functions**
- **Centre of mass**

…..and more…

Areas between curves

The area of a region bounded by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$\int_a^b [f(x) - g(x)]dx$$

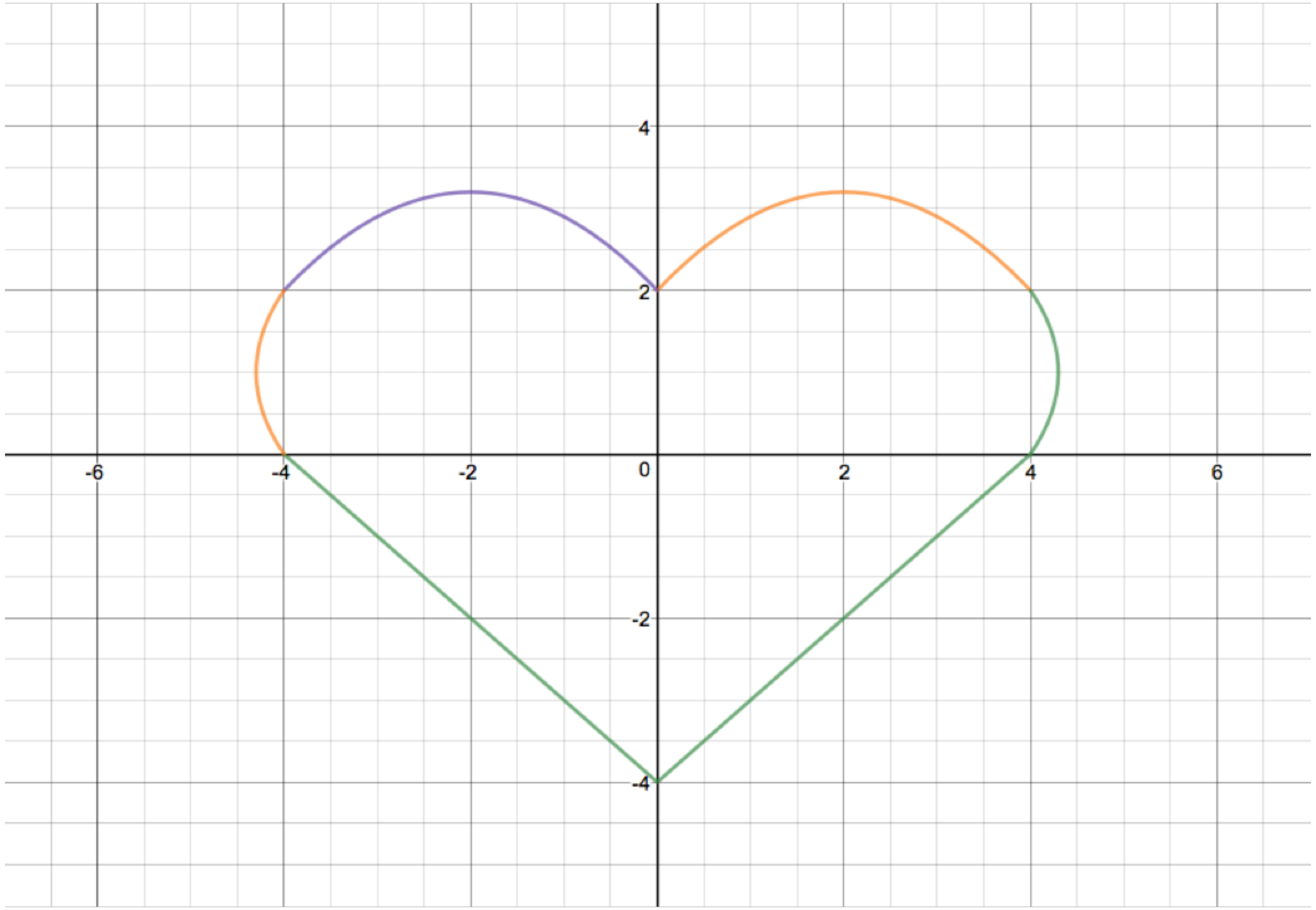
where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$.

Problems

1) Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.
2) Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.
3) Redo problem 2 by integrating with respect to y.
4) Find the total area A lying between the curves $y = \sin x$ and

   $y = \cos x$ from $x = 0$ to $x = 2\boldsymbol{\pi}$.

Compute the area enclosed by the heart-shaped curve (from February 14, 2017 midterm)



$y = |x| - 4$ if $-4 \le x \le 4$

$y = 0.3x(4 - x) + 2$ if $0 \le x \le 4$

$y = -0.3x(4 - x) + 2$ if $-4 \le x \le 0$

$x = -0.3y^2 + 0.6y + 4$ if $0 \le y \le 2$

$x = 0.3y^2 - 0.6y - 4$ if $0 \le y \le 2$