Science One Mathematics

Learning Outcomes

A. Limits

1. Explain (either in your own or by using a picture) the meaning of and give a rigorous definition of the expressions $\lim_{x \to a} f(x) = L$, $\lim_{x \to a+} f(x) = L$, $\lim_{x \to a-} f(x) = L$, where *a* is a finite number and *L* is either a finite number or (positive or negative) infinity.
2. Explain (either in your own or by using a picture) the meaning of and give a rigorous definition of the expressions $\lim_{x \to +\infty} f(x) = L$, $\lim_{x \to -\infty} f(x) = L$, where *L* is either a finite number or (positive or negative) infinity.
3. Evaluate the above limits (including limits that contain a parameter) either graphically or algebraically, or determine the limit does not exist; if the limit is an indeterminate form ($0/0$, ∞/∞, $\infty-\infty$, $0\cdot\infty$, 0^{∞}, 1^{∞}, ∞^{0}) use appropriate techniques to evaluate it.
4. State the limit laws and apply them to evaluate limits.
5. State and apply the Squeeze theorem.
6. State and apply l'Hopital's rule, recognize when the rule does or does not apply.
7. Give a formal definition of continuity and explain in your own words and by using a picture what it means for a function to be continuous at a point and on an interval, give examples of functions that are either continuous or discontinuous at a point.
8. Give examples of model real-life quantities that can be modelled by either a continuous or discontinuous function.
9. Given a piecewise function containing parameters, determine the parameter value(s) that make the function continuous on a given interval.
10. State the Intermediate Value Theorem, explain in your words its assumption(s) and conclusion.
11. Give examples of functions that do not satisfy the Intermediate Value Theorem by virtue of their discontinuity either in the interior of an interval or at its endpoints.
12. Apply the Intermediate Value Theorem to prove the existence of a root of an equation and to estimate the value of the root.

B. Derivatives and Antiderivatives

1. Explain how to define the instantaneous rate of change of a function/slope of tangent line as a limit of a difference quotient, and argue why these concepts require the mathematical notion of limit.
2. Give a definition of derivative as a limit, both at a point and as a function; recognize and use different notations (Newton or Leibnitz notation) for the derivative.
3. Give a definition of the second (and higher order) derivative and its interpretation in terms of rate of change.

4. Using the limit definition, calculate the derivative of polynomial functions of degree 3 or less, of rational functions $p(x)/q(x)$ where p and q are polynomials of degree 3 or less, and of square root functions.
5. Given a curve of equation $y = f(x)$, compute the slope of and find the equation of a line tangent to the curve at a given point; solve geometrical problems involving tangent lines.
6. Give a definition and interpretation of velocity and acceleration of a moving object in terms of derivatives; given the graphs or equations of position, velocity and acceleration of a moving object as a function of time, identify which is which and compute instantaneous velocities and accelerations using derivatives.
7. Give a mathematical definition of antiderivative; compute antiderivatives of elementary functions (polynomial functions with any power, simple rational functions, simple trigonometric functions, and simple exponential functions).
8. Given the graph of a function f, sketch the graph of f' and f'', and vice versa.
9. Give a geometrical proof for why $\lim_{x \to 0} \sin(x)/x = 1$, and use this limit to prove that $(\sin x)' = \cos x$.
10. Give an empirical explanation for why $\lim_{x \to 0} (e^x - 1)/x = 1$, and use it to define the derivative of e^x, and extend the idea to any base.
11. Apply appropriate differentiation rules (individually or in combination with other rules) to compute derivatives analytically.
12. Using either the definition of derivative as a limit or appropriate differentiation rules, prove simple statements involving derivatives.
13. Define what it means for a function to be differentiable at a point; explain why differentiable functions are continuous; demonstrate, using an example, that continuous functions need not be differentiable.
14. Given a piecewise function containing parameters, determine the parameter value(s) that make the function differentiable at specific points.

C. Implicit differentiation and logarithmic differentiation

1. Given an implicit relationship between two variables x and y, find dy/dx by applying the technique of implicit differentiation.
2. Justify the technique of implicit differentiation using the Chain Rule.
3. Given the equation of a curve that does not represent a function (and has equation of the form $f(x,y) = g(x,y)$ where f and g are composed of elementary functions), find the equation of lines tangent to the curve.
4. Use implicit differentiation to find the derivative of $\arcsin(x)$ and $\arccos(x)$ and $\ln x$.
5. Use logarithmic differentiation to differentiate complicated products of functions and functions of the form $[f(x)]^{g(x)}$.

D. Applications of the Derivative

D.1 Approximations

D.1.1 Linear Approximation

1. using a graph, explain how to obtain the linear approximation of a function at a point;
2. compute the linear approximation of a given function at a given point (“center”);
3. use appropriate linear approximations to estimate a function value;
4. express linear approximation in differentials notation.

D.1.2 Taylor Polynomials

1. find the Taylor polynomial (of a given degree) of a function at a point (“center”); explain of the derivatives of the function relate to the coefficients of the polynomial;
2. explain how linear approximation relates to Taylor polynomials;
3. use Taylor polynomials to approximate function values;
4. discuss the error in the approximation of function values obtained from a Taylor polynomial, find estimates of the error in a Taylor polynomial approximation;
5. use Taylor polynomials to evaluate limits.

D.2 Rate of change Problems

1. Use derivatives to compute rates of change, and solve problems involving rates of change; in particular, solve related rates problems, that is, problems where two (or more) quantities are related to each other and one seeks the instantaneous rate of change of one quantity given the rate of change of the other quantity.

D.3 Shape of a curve

D.3.1 The Mean Value Theorem (MVT)

1. State the MVT, in particular list its assumptions and conclusion, explain the conclusion of the theorem using a graph.
2. Explain how the MVT connects the value of the derivative of a function at a point to the graph of the function over an interval.
3. Apply the MVT to construct simple proofs.

D.3.2 Increasing/decreasing functions and concavity

1. Find critical numbers of a function.
2. Determine the intervals of increase/decrease of a function.
3. State the first and second derivative tests; apply the tests to find the local/relative extreme values of a function.
4. Define the terms “concave up”, “concave down” and “inflection point”; explain the connection between the concavity of a function and the sign of its second derivative.
5. Given a function, identify its inflection points and intervals of concavity.

D.3.3 Curve Sketching

1. Sketch the graph of a function, indicating its domain, intercepts, asymptotes, intervals of increase/decrease and extreme values, intervals of concavity and inflection points.

D.3.4 Optimization

1. Find the absolute/global maximum and minimum of a function.
2. Solve applied optimization problems.

4. Differential Equations

1. Explain in your words: what a differential equation is, what it means for a function to be a solution, what the terms "general/particular solution" mean in the context of a differential equation, what it means for a function to be a solution to a differential equation with pre-defined conditions (initial conditions or boundary conditions).
2. Check whether a function is a solution to a given differential equation; check whether a function is a solution to a given differential equation with specific initial/boundary conditions.
3. Solve separable differential equations where the antidifferentiation step is straightforward (i.e. direct application of the definition of antiderivative); express the solution to an initial/boundary value problem in explicit form, whenever possible.
4. Model quantities that grow or decay at a rate proportional to the quantity size; solve the associated differential equation, using the properties of exponential and logarithmic functions to find the constants in the model and make predictions.
5. Given an autonomous differential equation, find its steady states or equilibrium solutions.
6. Given an autonomous differential equation and an initial condition, predict the long-term behaviour of the solution, sketch a qualitative behaviour of the solution.
7. Explain how Euler's method works from a mathematical perspective, apply the method to approximate solutions of simple differential equations.