Last topic: sequences, series, power series, Taylor series...

- **sequences**: $1,\ 1,\ \frac{1}{2},\ \frac{1}{6},\ \frac{1}{24},\ \frac{1}{120}, \ldots \quad (\ \frac{1}{n!}\ , \quad n = 0, 1, 2, 3, \ldots)$
  - how does the n-th term of a sequence behave as $n \to \infty$?

- **series** (infinite sums): $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots \quad (= e)$
  - its "partial sums" form a sequence:
    $1,\ 1 + 1 = 2,\ 1 + 1 + \frac{1}{2} = \frac{5}{2},\ 1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3}, \ldots$
  - what does it mean to add together infinitely many numbers?

- **power/Taylor series**: $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots$
  $(= e^x)$
  - such a series gives a function of x (or does it?)
  - which functions can be represented as series?
  - connect back to calculus: differentiate, integrate...

Sequences

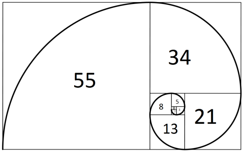
A **sequence** (or **infinite sequence**) is an ordered list of numbers with a first element, but no last element:

$$a_1, a_2, a_3, a_4, \cdots \quad = \quad \{a_n\}_{n=1}^{\infty} \quad = \{a_n\}$$

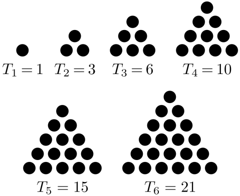
(that is, a function whose domain is the set of positive integers).

- $\{\frac{1}{n}\}_{n=1}^{\infty}$: $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5}, \ldots$
- $1,\ -\frac{1}{3},\ \frac{1}{9},\ -\frac{1}{27},\ \frac{1}{81},\ -\frac{1}{243}, \ldots$: $a_n = \left(-\frac{1}{3}\right)^n, \quad n = 0, 1, 2, \ldots$
- $-E_0,\ -\frac{E_0}{4},\ -\frac{E_0}{9},\ -\frac{E_0}{16}, \ldots$: $\{-\frac{E_0}{n^2}\}_{n=1}^{\infty}$, hydrogen energy levels
- $2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ \ldots$: $a_n =$ n-th prime number

- 1, 1, 2, 3, 5, 8, 13, . . . (*Fibonacci*)



  - — $a_1 = 1, a_2 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$
  - — in fact: $a_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$
- 2, 7, 1, 8, 2, 8, 1, 8, . . .: digits of $e = 2.7182818284\cdots$



- 1, 3, 6, 10, 15, 21, . . . (*triangular numbers*)
  - — $T_n = \sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$
- 1, 11, 21, 1211, 111221, 312211, 13112221, . . .
  - – hint: say it out loud

Convergence of sequences

Example: how does the n-th term of the sequence

$\frac{2}{2}, \frac{3}{4}, \frac{4}{6}, \frac{5}{8}, \frac{6}{10}, \frac{7}{12}, \frac{8}{14}, \frac{9}{16}, \frac{10}{18}, \dots$ behave as $n \to \infty$?

$a_n = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}$ tends toward $\frac{1}{2}$ as $n \to \infty$.

We say a sequence $\{a_n\}$ **converges** to the limit L, and write

$$\boxed{\lim_{n\to\infty} a_n = L} \quad \text{if:}$$

- "we can make a_n as close as we like to L by taking n large enough"; that is,
- for every $\epsilon > 0$, there is an integer N such that

$$n > N \implies |a_n - L| < \varepsilon \ .$$

If a sequence does not converge to a limit, we say it **diverges**.
(If $\lim_{n\to\infty} a_n = \pm\infty$ we may say the sequence *diverges to* $\pm\infty$.)

- limits of sequences obey all the same rules as limits of functions
- if $a_n = f(n)$ where $f = f(x)$ is a function on the real line and $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$

Determine if each sequence converges, and if so find its limit:

1. $\left\{ \frac{n^2}{n^2+n+1} \right\}$

2. $a_n = \cos\left(\frac{1}{n}\right), \quad a_n = \cos(n\pi), \quad a_n = \cos(2n\pi)$

3. $\{r^n\}_{n=1}^{\infty}: \quad r,\ r^2,\ r^3,\ r^4,\ r^5,\ \ldots$

4. $a_n = \ln(n+1) - \ln(n)$

5. $a_n = \left(1 + \frac{1}{n}\right)^n$

6. $\left\{ \frac{n!}{n^n} \right\}$

1. $\left\{\frac{n^2}{n^2+n+1}\right\}$: $\lim_{n\to\infty}\frac{n^2}{n^2+n+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}+\frac{1}{n^2}} = \frac{1}{1+0+0} = \boxed{1}$

2. $a_n = \cos\left(\frac{1}{n}\right)$, $a_n = \cos(n\pi)$, $a_n = \cos(2n\pi)$:
$\lim_{n\to\infty}\cos\left(\frac{1}{n}\right) = \cos(0) = \boxed{1}$,
$\{\cos(n\pi)\} = \{-1, 1, -1, 1, -1, 1, \dots\}$ $\boxed{\text{diverges}}$
$\{\cos(2n\pi)\} = \{1, 1, 1, 1, 1, 1, \dots\}$ converges to $\boxed{1}$

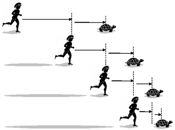
3. $\{r^n\}_{n=1}^{\infty}$: $r,\ r^2,\ r^3,\ r^4, r^5,\ \dots$
diverges if $r \le -1$ or $r > 1$, converges to $\boxed{1}$ if $r = 1$,
and if $|r| < 1$, $\lim_{n\to\infty} r^n = \boxed{0}$

4. $a_n = \ln(n+1) - \ln(n)$: $a_n = \ln\left(\frac{n+1}{n}\right) = \ln\left(1 + \frac{1}{n}\right)$ converges to $\ln(1) = \boxed{0}$

5. $a_n = \left(1 + \frac{1}{n}\right)^n$ $\lim_{n\to\infty}\ln(a_n) = \lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})}{\frac{1}{n}} = \lim_{n\to\infty}\frac{-\frac{1}{n^2}}{(1+\frac{1}{n})(-\frac{1}{n^2})}$
(l'Hôpital) $= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$, so $\lim_{n\to\infty} a_n = e^1 = \boxed{e}$

6. $\left\{\frac{n!}{n^n}\right\}$ $0 \le a_n = \left[\frac{n}{n}\,\frac{n-1}{n}\,\frac{n-2}{n}\cdots\frac{2}{n}\right]\frac{1}{n} \le \frac{1}{n}$, so by 'squeeze',
$\lim_{n\to\infty} a_n = \boxed{0}$.

Series



Can we add together together infinitely many numbers?

Try

$$1 \quad + \quad 1 \quad + \quad 1 \quad + \quad 1 \quad + \quad 1 \quad + \quad \cdots \quad ?$$

sums: $1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \cdots$

No! The running sum increases to ∞.

If A is twice as fast as T, the fraction of the initial gap to T that A makes up is:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots$$

sums: 0.500 0.750 0.875 0.938 0.969 0.984 0.992 $\cdots$

The running sum seems to settle down, tending, perhaps, toward 1.

A **series** (or **infinite series**) is an expression of the form

$$\sum_{j=1}^{\infty} a_j = a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$$

We make sense of a series by considering its **partial sums**:

$$s_n := \sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_{n-1} + a_n.$$

The partial sums themselves form a sequence of numbers:

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \quad s_4 = \cdots .$$

Let $s_n = a_1 + a_2 + \ldots + a_{n-1} + a_n$ be the n-th partial sum of a series $\sum_{j=1}^{\infty} a_j$. We say this series **converges** to the sum s if

$$\boxed{\lim_{n\to\infty} s_n = s}. \qquad \text{Then we write} \qquad \boxed{\sum_{j=1}^{\infty} a_j = s}.$$

If $\lim_{n\to\infty} s_n$ does *not* exist, we say the series **diverges**.

Examples

- $\sum_{k=1}^{\infty} k = 1 + 2 + 3 + 4 + 5 + \cdots$

$s_1 = 1,\ s_2 = 1 + 2 = 3,\ s_3 = 1 + 2 + 3 = 6,\ \ldots,\ s_n = \frac{1}{2}n(n+1)$

and $\lim_{n\to\infty} \frac{1}{2}n(n+1)$ does not exist $(= +\infty)$, so $\sum_{k=1}^{\infty} k$ <u>diverges</u>

- $\sum_{j=1}^{\infty}(-1)^j = -1 + 1 - 1 + 1 - 1 + 1 - \cdots$

sequence $\{s_n\}$ of partial sums $\{-1,\ 0,\ -1,\ 0,\ -1,\ \ldots\}$

does not converge, so $\sum_{j=1}^{\infty}(-1)^j$ <u>diverges</u>

These two series cannot possibly converge for a simple reason: the terms (the amount we add at each step) don't tend to zero!

Theorem: if $\sum\limits_{j=1}^{\infty} a_j$ converges, then $\lim\limits_{j\to\infty} a_j = 0$.

Proof: $s_n = \sum_{j=1}^{n} a_n$ (n-th partial sum). Then $s_n - s_{n-1} = a_n$, so $\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$. $\square$

We can rephrase this as a simple "test" for convergence of a series:

If $\lim\limits_{k\to\infty} a_k \neq 0$ (or does not exist), then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Example: $\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$: $\lim\limits_{k\to\infty} \frac{k(k+2)}{(k+3)^2} = \lim\limits_{k\to\infty} \frac{k^2+2k}{k^2+6k+9}$

$= \lim\limits_{k\to\infty} \frac{1+\frac{2}{k}}{1+\frac{6}{k}+\frac{9}{k^2}} = 1 \neq 0$, so this series diverges.

Warning: this test does not work in the opposite direction! There are series whose terms go to zero, but the series still fails to converge. An example is the "harmonic series"

$\sum_{j=1}^{\infty} \frac{1}{j} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ (more on this later...)

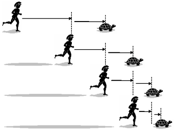
Geometric Series

A **geometric series** (with "common ratio" r) is

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots \quad = \sum_{j=1}^{\infty} ar^{j-1} \quad = \sum_{j=0}^{\infty} ar^j$$

Examples we've already seen today:

- $r = 1$, $a = 1$: $\sum_{j=0}^{\infty} 1 \cdot (1)^j = 1 + 1 + 1 + \cdots$ diverges
- $r = \frac{1}{2}$, $a = \frac{1}{2}$: $\sum_{j=1}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)^{j-1} = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$  $= 1$??

For which values of r does the geometric series

$$a+ar+ar^2+ar^3+ar^4+\cdots = a\left(1+r+r^2+r^3+r^4+\cdots\right) = \sum_{j=1}^{\infty} ar^{j-1}$$

converge? And to what sum?

First test: is $\lim_{j\to\infty} ar^{j-1} = 0$? Only if $-1 < r < 1$.

If $-1 < r < 1$, we compute the partial sums (blackboard):

$$s_n = \sum_{j=1}^{n} ar^{j-1} = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

and so $\lim_{n\to\infty} s_n = \frac{a}{1-r}$. Summary:

The geometric series $\sum_{j=1}^{\infty} ar^{j-1} = a+ar+ar^2+ar^3+ar^4+\cdots$

- converges, if $|r| < 1$, with $\sum_{j=1}^{\infty} ar^{j-1} = \frac{a}{1-r}$
- diverges, if $|r| \geq 1$