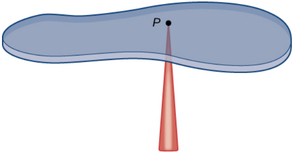
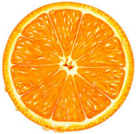
Application of Integration: Centre of Mass

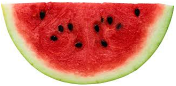
Goal: compute the **centre of mass** of a *lamina* (thin, flat plate)



i.e. the point on which it balances horizontally.

For example:





Centre of mass of 1D objects

First: what is the centre of mass of several point masses on a line?
If mass m_k sits at position x_k:

$$\boxed{\bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k} = \frac{M}{m}} = \frac{\textbf{moment (about } x=0)}{\text{total mass}}$$

Next: what is the centre of mass of a continuous 1D object (wire, rod) $a \leq x \leq b$ with given **linear density** (mass/unit length) $\rho(x)$?

$$\boxed{\bar{x} = \frac{\int_a^b x\rho(x)dx}{\int_a^b \rho(x)dx} = \frac{M}{m}} = \frac{\text{moment (about } x=0)}{\text{total mass}}$$

Example: Find the centre of mass of a wire $0 \leq x \leq L$ with (linear) density $\rho(x) = k\ x$:

$m = \int_0^L \rho(x)dx = k \int_0^L xdx = k\frac{x^2}{2}|_0^L = \frac{k}{2}L^2$

$M = \int_0^L x\rho(x)dx = k \int_0^L x^2dx = k\frac{x^3}{3}|_0^L = \frac{k}{3}L^3$

$\bar{x} = \frac{M}{m} = \frac{\frac{k}{3}L^3}{\frac{k}{2}L^2} = \boxed{\frac{2}{3}L}$

Centre of mass of 2D lamina

First: what is the centre of mass of several point masses in a plane?
If mass m_k sits at position (x_k, y_k): $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k} = \frac{M_y}{m}, \qquad \bar{y} = \frac{\sum_{k=1}^{n} m_k y_k}{\sum_{k=1}^{n} m_k} = \frac{M_x}{m}$$

Next: what is the centre of mass $(\bar{x}, \bar{y})$ of a 2D lamina *of constant density* whose shape is the region below $y = f(x)$, $a \le x \le b$?

$$\bar{x} = \frac{M_y}{m} = \frac{\rho \int_a^b x f(x) dx}{\rho \int_a^b f(x) dx} = \frac{\int_a^b x f(x) dx}{\int_a^b f(x) dx} = \frac{1}{A} \int_a^b x f(x) dx$$

$$\bar{y} = \frac{M_x}{m} = \frac{\rho \int_a^b \frac{1}{2}[f(x)]^2 dx}{\rho \int_a^b f(x) dx} = \frac{\int_a^b \frac{1}{2}[f(x)]^2 dx}{\int_a^b f(x) dx} = \frac{1}{A} \int_a^b \frac{1}{2}[f(x)]^2 dx$$

The constant density *cancels*. We also call $(\bar{x}, \bar{y})$ the **centroid**.

Example: Find the centroid of $\{0 \le y \le 1 - x^2,\ 0 \le x \le 1\}$:

$M = A = \int_0^1 (1 - x^2) dx = (x - \frac{1}{3}x^3)|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$

$M_y = \int_0^1 x(1 - x^2) dx = (\frac{1}{2}x^2 - \frac{1}{4}x^4)|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, $\bar{x} = \frac{1}{4} / \frac{2}{3} = \frac{3}{8}$

$M_x = \int_0^1 \frac{1}{2}(1 - x^2)^2 dx = \frac{1}{2} \int_0^1 (1 - 2x^2 + x^4) dx = \frac{4}{15}$, $\bar{y} = \frac{4/15}{2/3} = \frac{2}{5}$

Find the centroid of a $\frac{1}{4}$-disk.

- put it in the first quadrant: $0 \leq x \leq R, \;\; 0 \leq y \leq \sqrt{R^2 - x^2}$
- by symmetry, $\bar{x} = \bar{y}$
- $A = \frac{1}{4}\pi R^2$
- $\bar{x} = \frac{1}{A}\int_0^R x\sqrt{R^2 - x^2}dx = \frac{4}{\pi R^2}\left(-\frac{1}{3}(R^2 - x^2)^{\frac{3}{2}}\right)\Big|_0^R = \boxed{\frac{4R}{3\pi}}$
- double-check: $\quad \bar{y} = \frac{1}{A}\int_0^R \frac{1}{2}(\sqrt{R^2 - x^2})^2 dx$

$$= \frac{2}{\pi R^2}\int_0^R (R^2 - x^2)dx = \frac{2}{\pi R^2}(R^3 - \frac{R^3}{3}) = \frac{4R}{3\pi}$$

A little more on centroids

- for a region *between two graphs* $\{g(x) \le y \le f(x),\ a \le x \le b\}$

$$\boxed{\bar{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))dx, \quad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)dx}$$

- solids of revolution revisited ...

Pappus's Theorem: if a region of area A in the plane is rotated about a line L not intersecting it, the resulting volume is

$$\boxed{V = 2\pi\bar{r}\,A, \quad \bar{r} = \text{distance from centroid to } L}$$

Example: Use Pappus to find the volume of a doughnut (torus).

A doughnut is obtained by rotating a disk of radius r about a line a distance $R > r$ away from its centre. Pappus says:
$V = 2\pi R(\pi r^2) = \boxed{2\pi^2 R r^2}$. (Fun: do this using "shells".)