Science One Math

March 11, 2019

Applications of Integration

- Computing areas
- Computing changes
- Computing volumes
- Computing probabilities
- Locating centre of mass of a lamina

Key ideas ➥ “**slicing, approximating, adding** infinite contributions”

Today: Using integration to compute work done by a non constant force

From your Physics class (last Wednesday)

$\Delta U = -\int_i^f \vec{F} \cdot \overrightarrow{ds}$ change in potential energy (work done by $\vec{F}$)

For a conservative force (when work is independent of path), we can define a potential U such that

$$F_s = -\frac{dU}{ds}$$

This is the **fundamental theorem of calculus!**

Fundamental Theorem of Calculus is fundamental in Physics too!

FTC tells us $\int_a^b F(x)dx = U(b) - U(a)$ where $U(x) = \int F(x)dx + C$.

FTC also tells us $F(x) = \frac{dU}{dx}$ where $U(x) = \int_a^x F(t)dt$.

We call $U(x)$ the **potential energy**, then $\int_a^b F(x)dx$ is **Work**

Convention:

$F(x) = \frac{dU}{dx}$ for an *external* force (exerted *on* object)

$F(x) = -\frac{dU}{dx}$ for a force exerted *by* the potential energy

How to derive $\int_i^f \vec{F} \cdot \overrightarrow{ds}$ mathematically

(for straight paths)

If $\vec{F}$ is constant along the path $\Rightarrow$ **basic definition of work**

$$W = \vec{F} \cdot \vec{s} = F_s \Delta s$$

(work done by a **constant force** $\vec{F}$ acting on a particle that moves along displacement $\vec{s}$)

How to derive $\int_i^f \vec{F} \cdot \overrightarrow{ds}$ mathematically
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If $\vec{F}$ changes along the path $\Rightarrow$ use Calculus!

- slice path into segments
- assume force is constant along each segment $\Rightarrow$ compute work done by constant force to move particle over the segment $\Delta W = \vec{F} \cdot \overrightarrow{\Delta s} = F_s \Delta s$
- add up all contributions to work
- take the limit for an infinite number of segments $\Rightarrow$ definite integral $\int_i^f \vec{F} \cdot \overrightarrow{ds}$

Computing work done by a non constant force

- **"slice"** path into *n* segments of length Δx

  k-th segment is $[x_k, x_{k+1}]$

- **approximate** force by a constant on each segment (possibly different for each segment)

  let $F(x_k^*)$ be force component along *k*-th segment, for $x_k \le x_k^* \le x_{k+1}$

- **compute work** done by (constant) force on each segment of path

  $\Delta W = F(x_k^*)\Delta x$

- **add up** small amounts of work $\Rightarrow$ **Riemann Sum**

  $\sum_{k=1}^{n} F(x_k^*)\Delta x$

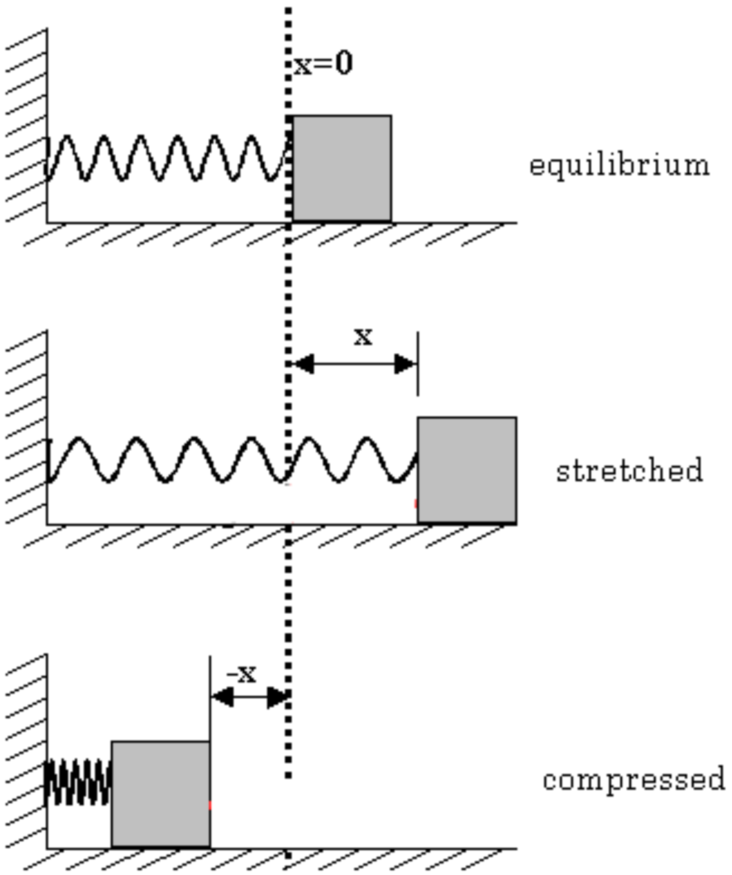
- take the **limit for $\boldsymbol{n}$** $\to \infty \Rightarrow$ a definite integral

  $W = \lim_{n\to\infty} \sum_{k=1}^{n} F(x_k^*)\Delta x = \int_a^b F(x)dx$

A few examples of non constant forces:

- elastic force
- electric force
- gravity on a point-like object of varying mass
- gravity on a distributed mass
- force exerted by (on) a gas during gas expansion

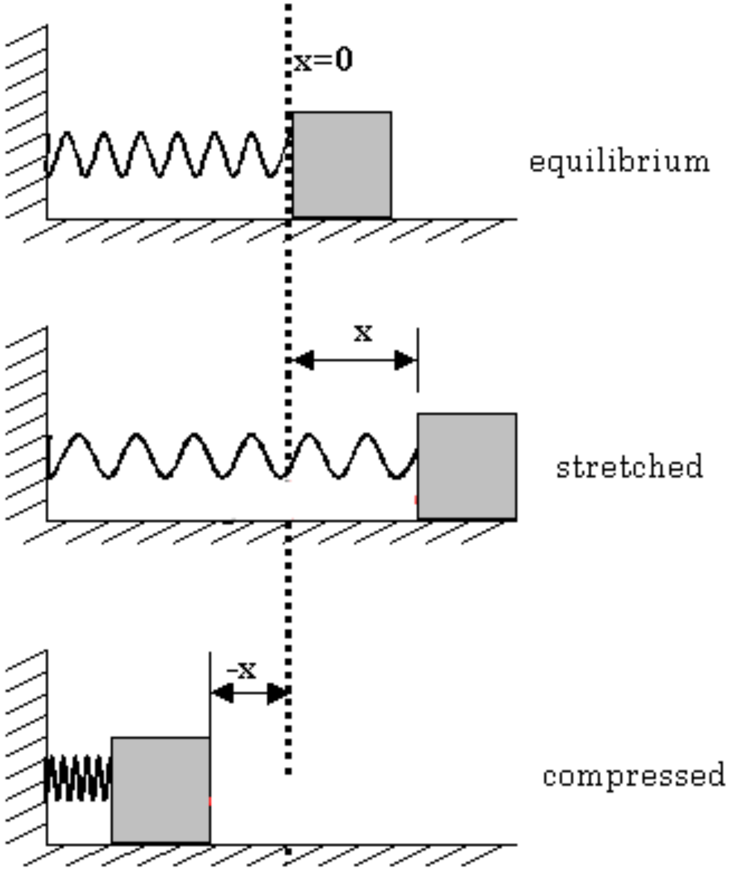
Work done on a spring



Hooke's law : force required to keep a spring compressed or stretched a distance x is *proportional* to x.

Note: x is measured from the natural length of spring.

Work done on a spring



Problem: Compute the work done **on the spring** to compress it by X.

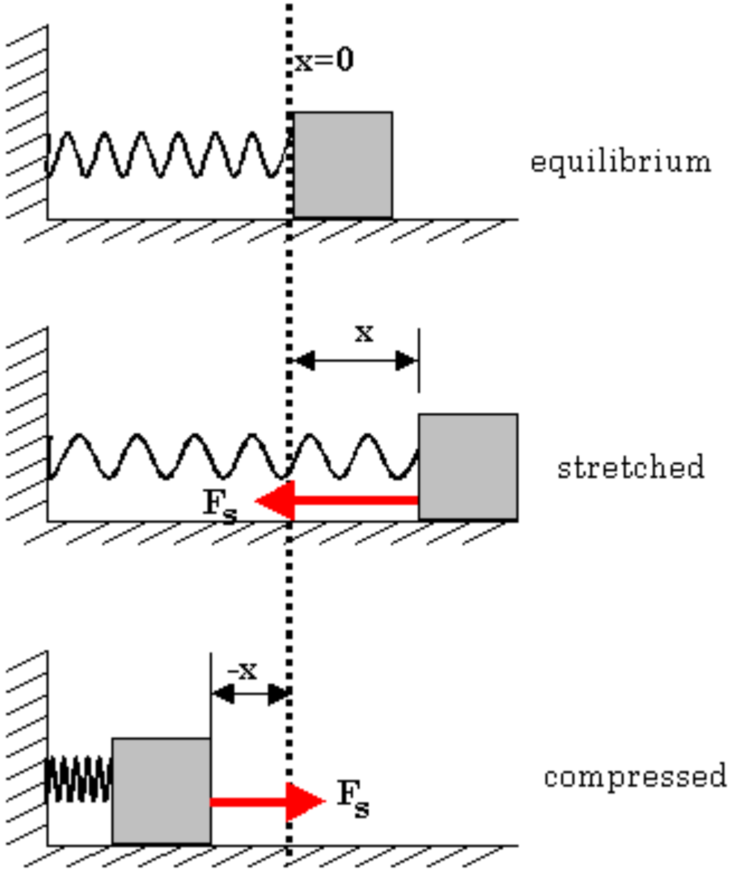
- "slice path", each segment is Δx long
- approximate force as a constant on each segment, on i-th segment, consider force of magnitude $F_i = kx_i$

- compute work to compress by Δx,
  Recall: Force exerted **on spring** is in the *same direction* as displacement, $\vec{F_i} \cdot \vec{\Delta x} = F_i \Delta x \cos(0) = F_i \Delta x$
  $$\Rightarrow \Delta W_i = kx_i \Delta x$$

- add up all contributions and take a limit $\Rightarrow$ definite integral

$$W = \int_0^{-X} kx\, dx = \frac{1}{2} k\, X^2$$

Work done by a spring



Problem: Compute work done **by the spring** when compressed by X.

- approximate force as a constant on each segment, on i-th segment, consider force of magnitude $F_i = kx_i$
- compute work when compressed by Δx,

Recall: Force exerted **by spring** is *opposite* to displacement, $\overrightarrow{F_i} \cdot \overrightarrow{\Delta x} = F_i \, \Delta x \cos(\pi) = -F_i \Delta x$

$\Rightarrow \Delta W_i = -kx_i \Delta x$

- add up all contributions and take a limit $\Rightarrow$ definite integral

$$W = -\int_0^{-X} kx\,dx = -\frac{1}{2}\,k\,X^2$$

Work done on a point charge

Problem: Find the electric potential energy between two charges a distance r apart.

Recall: When a conservative force acts on a particle that moves from a to b, the change in potential energy is the negative work done by conservative force, $U_b - U_a = -W$.

Strategy: Compute work done on q_1 by the electric force exerted by a second (stationary) charge q_2 when q_1 moves from very far (∞) to r.

Work done on a point charge

Problem: Find the electric potential energy between two charges a distance r apart.

- "slice" path, each segment is Δr long
- approximate force, on the i-th segment consider $F_i = \frac{kq_1q_2}{(r_i)^2}$
- compute work done to move q_1 by Δr

Recall: Electric force is in the same direction as the displacement

$$\Rightarrow \Delta W_i = \overrightarrow{F_i} \cdot \overrightarrow{\Delta r} = F_i \Delta r_i = \frac{kq_1q_2}{(r_i)^2} \Delta r_i$$

Add up all contributions and take a limit $\Rightarrow$ $W = \int_{\infty}^{r} \frac{kq_1q_2}{z^2} dz$ improper integral

$$\Delta U = -\int_{\infty}^{r} \frac{kq_1q_2}{z^2} dz = -\lim_{R\to\infty} -\frac{kq_1q_2}{z}\Big|_R^r = \frac{kq_1q_2}{r} - \lim_{R\to\infty} \frac{kq_1q_2}{R} = \frac{kq_1q_2}{r}$$

A leaky bucket…

A 2 kg bucket and a light rope are used to draw water from a well that is 40 m deep. The bucket is filled with 20 kg of water and is pulled up at 0.5 m/s, but water leaks out of a hole in the bucket at 0.1 kg/s. Find the work done in pulling the bucket to the top of the well.

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Force acting on bucket changes as water leaks out ⇒ need to integrate!

Solution

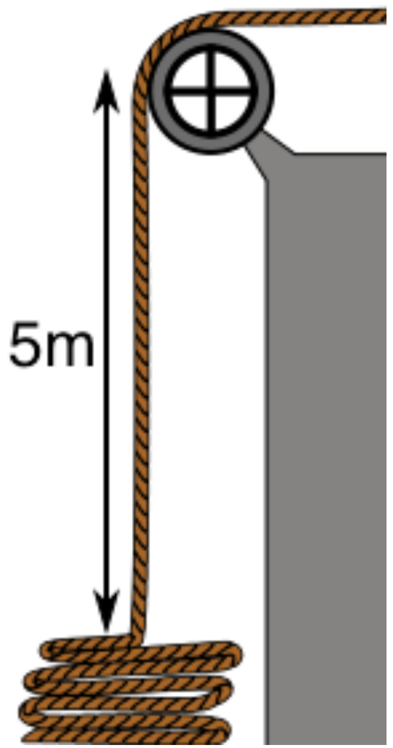
- slice up path into segments Δy long
- on the i-th segment consider $F_i = m_i g$
- work to lift water by Δy is $\Delta W = m_i g \Delta y \Rightarrow m_i = m(y_i)$, m is a function of y

$$\frac{dm}{dy} = -\frac{0.1}{0.5},\; m(0) = 20 \;\Rightarrow\; m(y) = -0.2y + 20$$

$$W_{water} = \int_0^{40} (-0.2y + 20)g\, dy \qquad W_{bucket} = mg \cdot 40 = 80g \quad (\text{constant } m)$$

$$W_{total} = W_{water} + W_{bucket}$$

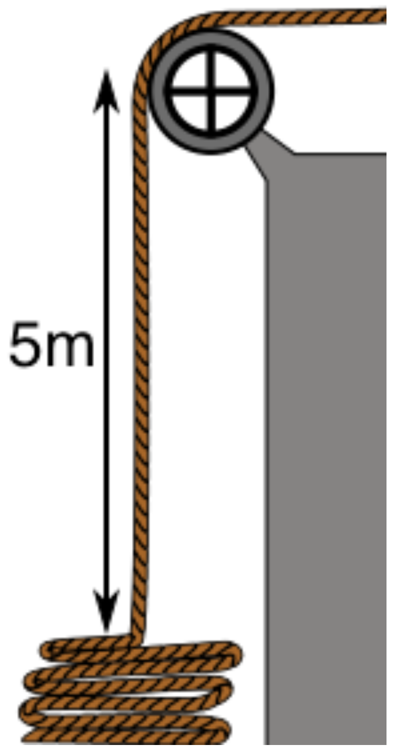
A heavy rope…



A 10-m long rope of density 2 kg/m is hanging from a wall which is 5 m high (so 5 m of rope runs down the length of the wall and the remaining 5 m is coiled at the bottom of the wall).

How much work (in J) is required to pull the rope to the top of the wall? Let g be the acceleration due to gravity.

A heavy rope…



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Rope has distributed mass, NOT point-like object

☛portion of rope near the top undergoes small displacement,

☛portion of rope near the ground undergoes bigger displacement

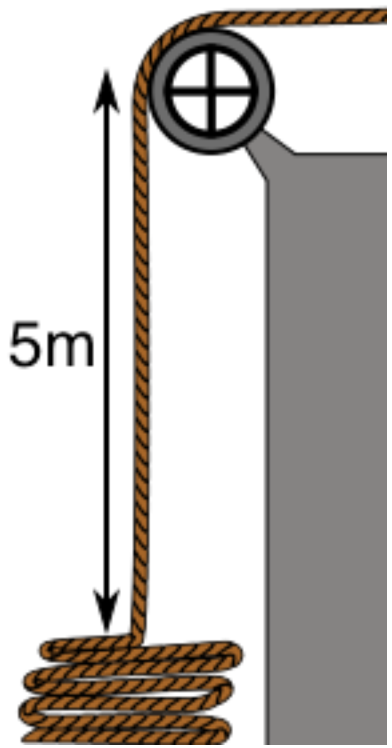
Mass is distributed uniformly along rope ⇒ force is constant

Displacement changes ⇒ need to integrate!

Strategy:

- **"slice" rope into small segments of mass Δm.**
- **Compute work to lift each segment to the top of wall**

A heavy rope…



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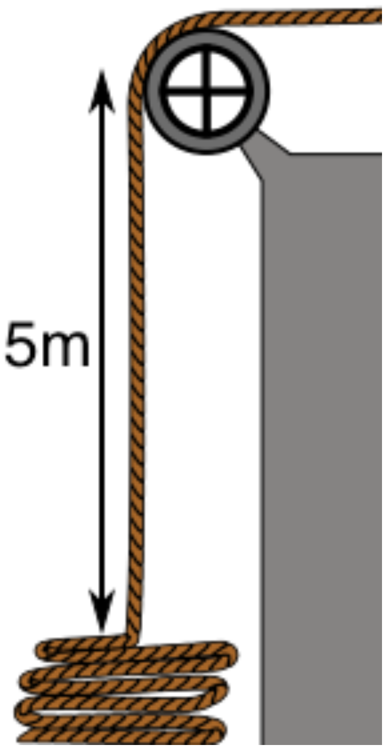
What's the work ΔW to lift a segment of rope Δy long from a height y to the top of the wall?

A. $\Delta W = (2\Delta y)g\, y$
B. $\Delta W = (2\Delta y)g(5 - y)$
C. $\Delta W = (2\Delta y)g(10 - y)$
D. $\Delta W = (2\Delta y)g(5 + y)$
E. $\Delta W = (2\Delta y)g(10 + y)$

A heavy rope…



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What's the work ΔW to lift a segment of rope Δy long from a height y to the top of the wall?

A. $\Delta W = (2\Delta y)gy$

B. $\boldsymbol{\Delta W = (2\Delta y)g(5-y)}$

C. $\Delta W = (2\Delta y)g(10-y)$

D. $\Delta W = (2\Delta y)g(5+y)$

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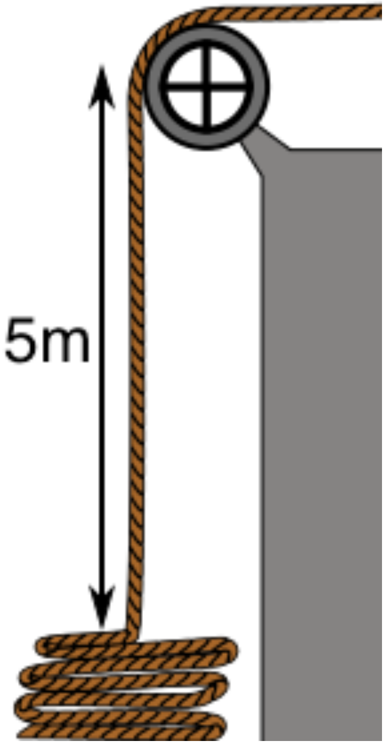
A heavy rope…



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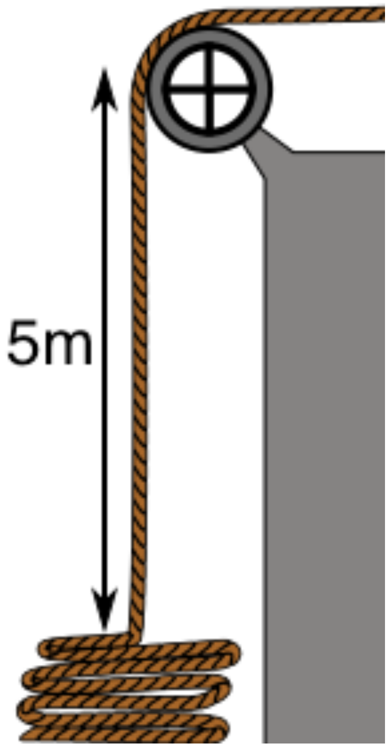
How much work (in J) is required to pull the rope to the top of the wall? Let g be the acceleration due to gravity.

A. $W = \int_0^5 2g(5-y)dy$

B. $W = \int_0^{10} 2g(5-y)dy$

C. $W = \int_0^5 2g(5-y)dy + 50g$

A heavy rope…



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A segment (of hanging rope) at height y moves a distance $(5-y)$

A segment (of coiled rope) moves a distance of 5 m (constant displacement, no need to integrate)

total work $W = \int_0^5 2g(5-y)\,dy + 2 \cdot 5 \cdot g \cdot 5$

force   displacement

Building a pile of sand…

How much work must be done in producing a conical heap of sand of base radius R and height H? Let ρ be the density of mass (kg/m^3). You may assume that all the sand is taken from the surface of the earth (that is, from height 0).

Building a pile of sand...

How much work must be done in producing a conical heap of sand of base radius R and height H? Let ρ be the density of mass (kg/m^3). You may assume that all the sand is taken from the surface of the earth (that is, from height 0).

No work is done when moving sand horizontally.
☛ less mass at the top of pile compared to the bottom
☛ sand at the top travels higher than sand at the bottom

Both mass (force) and displacement change $\Rightarrow$ Integrate!

Strategy:
- **"slice" pile into horizontal layers of mass Δm**
- **compute work done to lift each layer to its current height**

Building a pile of sand

How much work must be done in producing a conical heap of sand of base radius R and height H? Let ρ be the density of mass (kg/m^3). You may assume that all the sand is taken from the surface of the earth (that is, from height 0).

Work to lift a layer of mass Δm up a height y from the ground

force displacement

$$\Delta W = (\Delta m_{layer} \cdot g) \cdot y$$

$$\Delta m_{layer} = \rho \cdot \Delta V_{layer}$$

$$\Delta V_{layer} = \pi r^2 \Delta y = \pi \left(R - \frac{R}{H} y\right)^2 \Delta y$$

$$W = \int_0^H \rho \, g \, \pi \left(R - \frac{R}{H} y\right)^2 y \, dy$$

Digging a well

Consider two workers digging a well. How deep should the first worker dig so that each does the **same amount of work**?

Assume the well does not get any wider or narrower as the workers dig.

A. The first worker should dig to a depth $D/2$
B. The first worker should dig to a depth $D/3$
C. The first worker should dig to a depth $2D/3$
D. The first worker should dig to a depth $3D/4$
E. The first worker should dig to a depth $D/\sqrt{2}$

Digging a hole

Consider two workers digging a hole. How deep should the first worker dig so that each does the same amount of work?

ρ = density of the dirt (constant)

D = depth of the well (fixed)

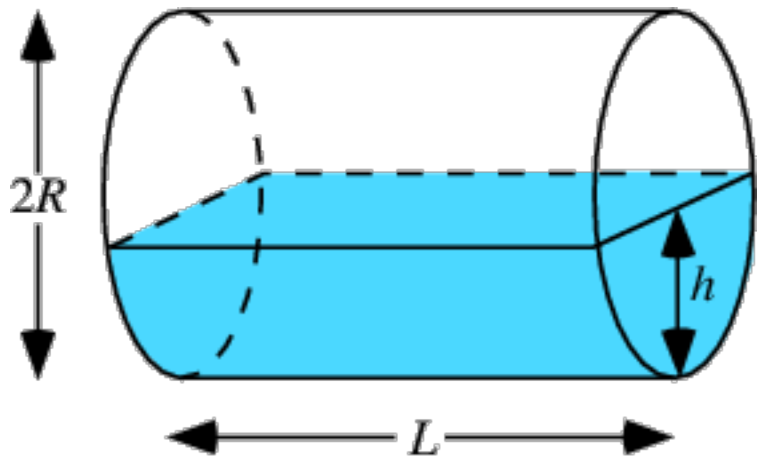
A = cross-sectional area of the well (constant)

$$W_{tot} = \int_0^D \rho\, A\, g\;\, y\, dy = \rho A g\, \frac{D^2}{2}$$

Let z be the depth of the hole the first worker, must solve

$$\int_0^z \rho\, A\, g\, y dy = \frac{1}{2}\, \rho A g\, \frac{D^2}{2} \quad \Rightarrow\; z = \frac{D}{\sqrt{2}}.$$

Pumping out fluid from a tank



A cylindrical tank with a length of L m and a radius of R m is on its side and half-full of gasoline. How much work is done to empty the tank through an outlet pipe at the top of the tank?

Let ρ be density of gasoline, and A be the cross-sectional area of a layer at height y,

A. $\int_0^{2R} \rho\, g\, A\,(2R - y)dy$

B. $\int_0^{2R} \rho\, g\, A\,(R - y)dy$

C. $\int_0^{R} \rho\, g\, A\,(2R - y)dy$

D. $\int_0^{R} \rho\, g\, A\, y\, dy$

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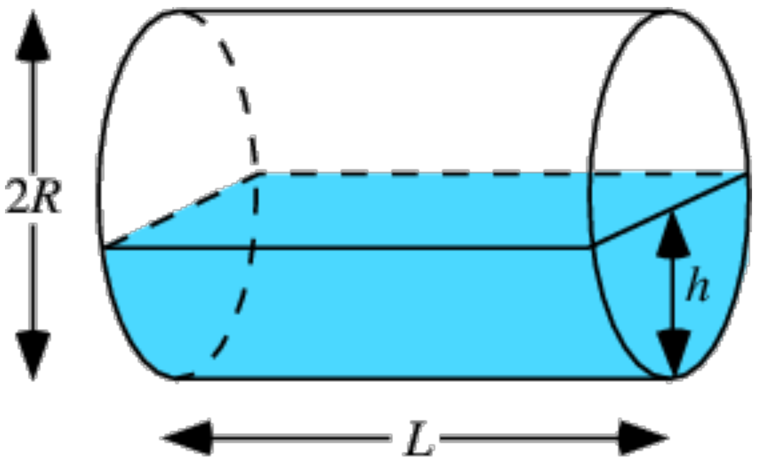

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B. $\int_0^{2R} \rho\, g\, A\,(R-y)dy$

C. $\boldsymbol{\int_0^{R} \rho\, g\, A\,(2R-y)dy}$

D. $\int_0^{R} \rho\, g\, A\, y\, dy$

E. $\int_0^{R} \rho\, g\, A\, dy$