Science One Math

March 20, 2019

One more test….let's start with an observation:

For a geometric series $\sum a_n = \sum_{n=0}^{\infty} a\, r^n$, the ratio of any two subsequent terms is constant

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$$

We need $|r| < 1$ for convergence.

The ratio test extends this idea.

The Ratio Test

- if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \boldsymbol{L < 1} \Rightarrow$ the series $\sum a_n$ **converges**
- if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \boldsymbol{L > 1}$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty \Rightarrow$ the series $\sum a_n$ **diverges**
- if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1 \Rightarrow$ the test is inconclusive.

Rationale: If the limit above exists, then the tail of the series behaves like a geometric series

Note: The ratio test works well when a_n involves exponentials and factorials.

Determine whether the following series converge or diverge:

- $\sum_{k=1}^{\infty} \frac{10^k}{k!}$
- $\sum_{n=1}^{\infty} \frac{n^n}{n!}$
- $\sum_{j=1}^{\infty} e^{-j}(j^2 + 4)$
- $\sum_{k=1}^{\infty} \frac{k}{3^k}$
- $\sum_{n=1}^{\infty} \frac{n!}{(2n)!}$

Warning: The ratio test may be inconclusive…

Simple example:

$1+1+1+1+\cdots$ we know it diverges but $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$

the ratio test cannot detect the divergence!

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$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ for any p, the ratio test does not feel the difference between $p = 2$ (convergence) and $p = 1$ (divergence).

The integral test is sharper! (but computing integrals may be hard!)