Science One Math

March 25, 2019

Series with negative terms

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If we replace all negative signs with +, we have a convergent geometric series.

Changing from a_n to $|a_n|$ increases the sum (replace negative numbers with positive numbers). The smaller series $\sum a_n$ will converge if the larger series $\sum |a_n|$ converges.

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Changing from a_n to $|a_n|$ increases the sum (replace negative numbers with positive numbers). The smaller series $\sum a_n$ will converge if the larger series $\sum |a_n|$ converges $\Rightarrow$ another test for convergence of $\sum a_n \ldots$

Test for Absolute Convergence:

If $\sum |a_n|$ converges, then $\sum a_n$ converges (absolutely).

Terminology: Absolute and Conditional Convergence

Definition of Absolute Convergence

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Test for Absolute Convergence:

- If $\sum |a_n|$ converges, then $\sum a_n$ converges (absolutely).

Note: if $\sum |a_n|$ diverges, $\sum a_n$ may or may not converge.

Examples

Determine whether the following series converge absolutely

- $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt{k^3}}$
- $\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2}$
- $\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{2p-1}$

Examples

Determine whether the following series converge absolutely

- $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt{k^3}}$ $\quad \sum_{k=1}^{\infty}|a_k| = \sum_{k=1}^{\infty}\frac{1}{k^{3/2}}$ p-series with p>1, **converges**

- $\sum_{n=1}^{\infty}\frac{\sin(n)}{n^2}$ (series with both positive and negative terms)

  $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\left|\frac{\sin(n)}{n^2}\right|$ **converges** by comparison with $\sum\frac{1}{n^2}$

  $$\frac{|\sin(n)|}{n^2} \leq \frac{1}{n^2}$$

- $\sum_{p=1}^{\infty}\frac{(-1)^{p-1}}{2p-1}$ $\quad \sum_{p=1}^{\infty}|a_p| = \sum_{p=1}^{\infty}\left|\frac{1}{2p-1}\right|$ **diverges** by comparison with $\sum\frac{1}{2p}$

Special case: Alternating series

Signs strictly alternate $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots$

This is an alternating harmonic series. We know it doesn't converge absolutely. **Does it converge conditionally?**

Special case: Alternating series

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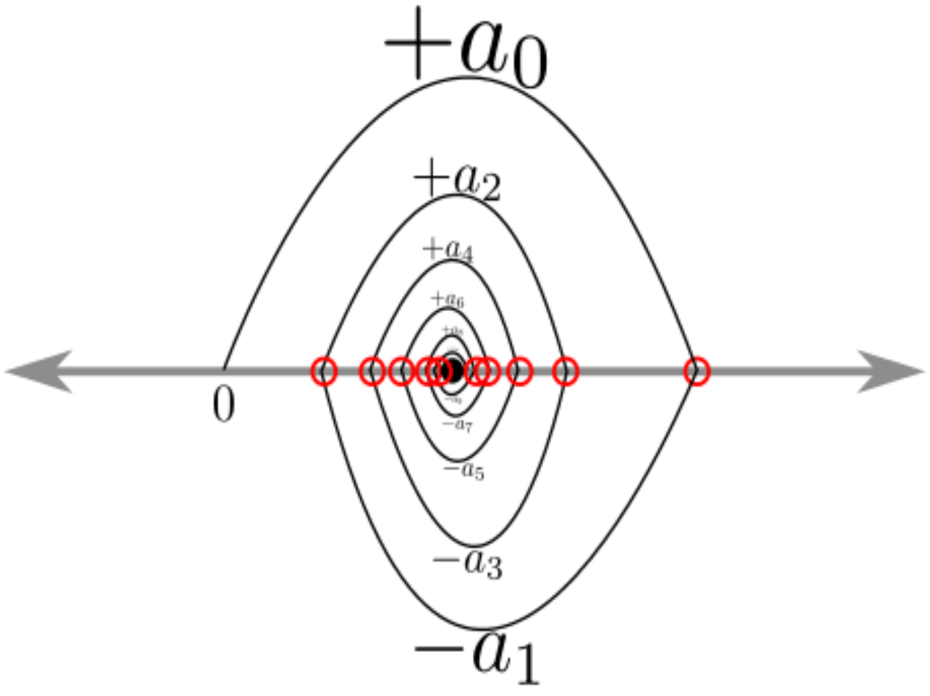
This is an alternating harmonic series. We know it doesn't converge absolutely. Does it converge conditionally? Look at the behaviour of the partial sums!

Consider
$\sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + a_4 \ldots$

where $a_{n+1} \leq a_n$ (decreasing sequence)

Intuitively: if the terms are alternating, decreasing, and go to zero, then the partial sums approaches a finite number
$\Rightarrow$ series converges



Alternating Series Test

If $\sum(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \cdots$ (with $a_n > 0$) is such that

- $a_{n+1} \leq a_n$ (decreasing sequence)
- $\lim_{n\to\infty} a_n = 0$

then the series $\sum(-1)^{n+1}a_n$ converges.

Examples

Determine if the following series converge

- $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

- $\sum_{n=1}^{\infty} \cos(n\pi) \frac{1}{2^n}$

- $\sum_{n=2}^{\infty} (-1)^n \frac{e^n}{n^5}$

- $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$

Examples

- $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ alternating harmonic series, converges
- $\sum_{n=1}^{\infty} \cos(n\pi) \frac{1}{2^n}$ alternating geometric series, converges
- $\sum_{n=2}^{\infty} (-1)^n \frac{e^n}{n^5}$ diverges because $\lim_{n \to \infty} \frac{e^n}{n^5} = \infty$
- $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}$ converges because $\lim_{n \to \infty} \frac{n!}{n^n} = 0$ and

$$\frac{(n+1)!}{(n+1)^{n+1}} = \frac{n!}{(n+1)^n} \leq \frac{n!}{n^n}$$

The algebra of convergent series

Can a convergent series be manipulated as a finite sum?

Yes, if it converges absolutely, otherwise no!

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The delicacy of conditionally convergent series

If a series **converges only conditionally**, the order of the terms is important.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} \cdots = \ln 2$$ *(see next week)*

Rearrange

$$\left(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} \cdots\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{12} \cdots\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{20} \cdots\right)$$

$$\underbrace{\left(1 - \sum \left(\frac{1}{2}\right)^n\right)}_{\to 0} + \frac{1}{3}\underbrace{\left(1 - \sum \left(\frac{1}{2}\right)^n\right)}_{\to 0} + \frac{1}{5}\underbrace{\left(1 - \sum \left(\frac{1}{2}\right)^n\right)}_{\to 0} + \cdots$$ we get $0 = \ln 2$ (!)

What have infinite series to do with calculus?

Convergent infinite series can be used to define functions!

Recall definition of a function: a function is a “rule” for assigning to each input value (x-value) a single output value (y-value).

A convergent series converges to its sum. If we changed the numbers in the series, the sum of the series is likely to change.

Numbers in series as “input” → sum of series as “output”

Our goal: go back from numbers to functions!

Which convergent series has a well-known sum? ➥ Geometric series

Which of these are **geometric series**?

i) $3 + 3 \cdot \frac{2}{3} + 3 \cdot \frac{4}{9} + 3 \cdot \frac{8}{27} + \cdots$

ii) $2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots$

iii) $1 + a + a^2 + a^3 + \cdots$ where a is a constant

iv) $1 - \frac{1}{2}(b-2) + \frac{1}{4}(b-2)^2 - \frac{1}{8}(b-2)^3 \ldots$ where b is a constant

A) *i*)
B) *i*) and *ii*)
C) All of them
D) None of them

Which of these are **geometric series**?

i) $3 + 3 \cdot \frac{2}{3} + 3 \cdot \frac{4}{9} + 3 \cdot \frac{8}{27} + \cdots$

ii) $2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \cdots$

iii) $1 + a + a^2 + a^3 + \cdots$ where *a* is a constant

iv) $1 - \frac{1}{2}(b-2) + \frac{1}{4}(b-2)^2 - \frac{1}{8}(b-2)^3 \ldots$ where *b* is a constant

A) *i*)

B) *i*) and *ii*)

C) All of them $\boldsymbol{a}$ **and** $\boldsymbol{b}$ **are PARAMETERS**

D) None of them

Power Series

If we treat the *parameter* as a *variable* x, we have a **power series**

e.g,

$$1 + x + x^2 + x^3 + \cdots$$

or

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 + \frac{1}{8}(x-2)^3 + \cdots$$

Think of these as “infinite polynomials”.

Power Series

A **power series** about a is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where a is fixed number, called ***centre*** of the series.

This is also called a power series in $(x-a)$.

Example: $1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ is centred at 0.

What is the centre of the series $\sum_{n=0}^{\infty} n^3 (2x-3)^n$?

$A)\ a = 2/3$

$B)\ a = 3/2$

$C)\ a = 2$

$D)\ a = 3$

E) None of the above

What is the centre of the series $\sum_{n=0}^{\infty} n^3 (2x-3)^n$?

A) $a = 2/3$

B) $\boldsymbol{a = 3/2}$

C) $a = 2$

D) $a = 3$

E) None of the above

To find the centre a of a power series, we want series in the form $\sum_{n=0}^{\infty} c_n (x-a)^n$

Do power series converge?

Power series can be used to define a function only if the series converges.

For what value(s) of x does a power series converge?

➤ 3 possible cases: 1) convergence at a **point** (*centre*) *always*

2) convergence over an **interval** *sometimes*

(*interval of convergence*)

3) convergence for **all** $\boldsymbol{x}$ *sometimes*

E.g. $\sum_{n=0}^{\infty} x^n$ converges for $|x| < 1$ or $-1 < x < 1$.

Assumption: When working with power series, we'll consider only x-values for which the series converges.