More ODE

An ~~object~~ pumpkin, initially at rest, is released in a constant gravitational field, and subject to a drag force proportional to its speed.  Write an ODE IVP to model this. Then solve it.

$v(t)$ = velocity (downward) as a function of time:

$$m\frac{dv}{dt} = mg - kv, \quad v(0) = 0$$

- $\int \frac{dv}{g - \frac{k}{m}v} = \int dt \implies -\frac{m}{k}\ln(g - \frac{k}{m}v) = t + C$ (why no $|\cdot|$?)
- $v(0) = 0 \implies -\frac{m}{k}\ln(g) = C \implies \ln(g - \frac{k}{m}v) = -\frac{k}{m}t + \ln(g)$
- $\implies g - \frac{k}{m}v = e^{-\frac{k}{m}t}g \implies \boxed{v(t) = \frac{mg}{k}(1 - e^{-\frac{k}{m}t})}$
- as $t \to \infty$, $v(t) \to \frac{mg}{k}$ (terminal velocity)

ODE: Population Growth Examples

Let $P(t)$ be the population of ~~some species~~ ~~bats~~ reindeer as a function of time, with initial population $P(0) = P_0$.

Recall: if the growth rate is proportional to the population, $\frac{dP}{dt} = kP$, we get **exponential growth**: $P(t) = P_0 e^{kt}$.

There may also be limits to growth (eg. scarce resources). A simple model is the **logistic equation**:

$$\frac{dP}{dt} = kP(1 - P/K), \quad P(0) = P_0.$$

Interpretation: when P is small it grows exponentially, but as P increases, its growth rate slows down, vanishing when P reaches a "carrying capacity" value K.

Note that the independent variable t does not appear in the ODE. Such an ODE is called **autonomous**.

ODE: Qualitative Analysis

How do we solve a first order autonomous ODE $\boxed{\frac{dP}{dt} = f(P)}$?

It is separable, so we can solve it by integration:

$\int \frac{dP}{f(P)} = \int dt = t + C.$

But we can learn "how solutions behave" without doing any integrals! This is sometimes called **qualitative analysis**.

Exercise (blackboard): for the logistic equation

$$\frac{dP}{dt} = kP(1 - P/K), \quad P(0) = P_0.$$

1. find the constant solutions (**equilibria** or **steady-states**)
2. sketch $f(P) = kP(1 - P/K)$, and so find where $\frac{dP}{dt} > 0, \quad < 0$
3. sketch some solutions $P(t)$, and predict the fate $\lim_{t\to\infty} P(t)$.

Exact solution (next term!): $P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-kt}}$.

ODE: More Population Growth

Exercise (blackboard): perform the same "qualitative analysis" for the first-order, autonomous ODE problem

$$\frac{dP}{dt} = k(P - m)(1 - P/K), \quad P(0) = P_0, \qquad 0 < m < K.$$

How does this model differ from the logistic equation?

ODE: A Boundary Value Problem

Find the *wavefunction* $\psi(x)$ for a quantum particle of mass m at energy E confined to a (1D) box $0 < x < L$ given that it solves the (stationary) Schrödinger equation ($k = \frac{\sqrt{2mE}}{\hbar}$):

$$\frac{d^2\psi}{dx^2} = -k^2\psi, \qquad \psi(0) = 0, \;\; \psi(L) = 0.$$

- is this an IVP? No – it's a BVP.
- general solution (by method of 'clever guessing'):
  $\psi(x) = A\sin(kx) + B\cos(kx)$
- $0 = \psi(0) = A\cdot 0 + B\cdot 1 = B \implies B = 0 \implies \psi(x) = A\sin(kx)$
- $0 = \psi(L) = A\sin(kL) \implies$ either $A = 0$ ($\psi = 0$) or $\sin(kL) = 0$
- $\sin(kL) = 0 \implies kL = n\pi$ for some $n = 0, 1, 2, 3, \ldots$
  $\implies E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2\pi^2}{2mL^2}n^2$ (quantized energy levels!)
- the wavefunction for the n-th energy level is
  $\psi_n(x) = A_n\sin(\frac{n\pi}{L}x)$
  (we often choose A_n to normalize $\int_0^L(\psi_n(x))^2dx = 1$)