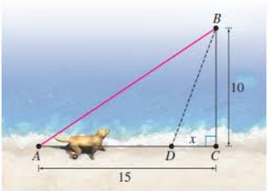
Optimization Problems

A ball floats 10 *m* from the water's edge. A dog named *Elvis*, who can run 10 m/s and swim 2 m/s, is 15 *m* along the shore from the point closest to the ball. Where should he enter the water in order to reach the ball in the shortest time?

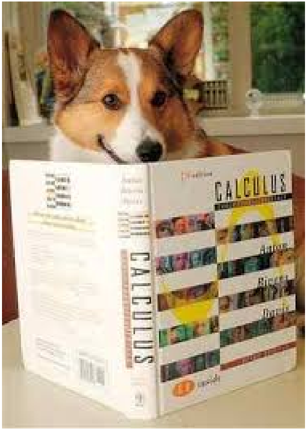


- x = distance of entry point from closest point (as in diagram)
- travel time: $T(x) = \frac{15-x}{10} + \frac{\sqrt{100+x^2}}{2}$, $\quad 0 \leq x \leq 15$
- endpoints: $T(0) = 6.5\ s$, $T(15) = \frac{\sqrt{325}}{2} \approx 9.01\ s$
- critical #s: $0 = T'(x) = -\frac{1}{10} + \frac{x}{2\sqrt{100+x^2}} \implies \sqrt{100+x^2} = 5x$

$$\implies 100 + x^2 = 25x^2 \implies x^2 = \frac{100}{24} \implies \boxed{x = \frac{10}{\sqrt{24}} \approx 2.04\ m}$$

and $T(\frac{10}{\sqrt{24}}) \approx 6.40\ s$, so this is indeed the optimal entry point

Apparently, some dogs can do calculus...



Optimization Problems

What rectangle of given area has the least perimeter? The most?

- draw a picture: $L =$ length, $W =$ width
- area $A = LW \implies W = \frac{A}{L}$
- perimeter $P = 2(L + W) \implies P(L) = 2(L + \frac{A}{L})$, $L > 0$
- critical #: $0 = P'(L) = 2(1 - \frac{A}{L^2}) \implies L = \sqrt{A}$, $P(\sqrt{A}) = 4\sqrt{A}$
- "endpoints": $\lim_{L \to 0+} P(L) = +\infty$, $\lim_{L \to +\infty} P(L) = +\infty$
- so (surprise!) $\boxed{\text{the square } L = W = \sqrt{A}}$ gives least perimeter
- the "most perimeter" problem has no solution

What rectangle of given perimeter has least area? Greatest?

Optimization Problems

General strategy for max/min problems:

1. **Read carefully**. What info is given? What must we compute?
2. **Draw a diagram!**
3. Assign variables to relevant quantities (*use good names*).
4. Find relations between the variables.
5. Reduce down to a **function of one variable to maximize/minimize**. What is its domain?
6. Now do the calculus: find critical numbers, and “endpoints”, and compare the values.
7. **Quick check**: did you indeed find the min/max? Right units?

Optimization Problems

Solve problem number (Bamfield # (mod 4)). Guess first, then "do the calculus". Find:

1. the max/min values of the product of 2 numbers whose squares sum to 1.
2. the max/min values of the distance from the origin of a point (x, y) on the hyperbola $xy = 1$.
3. the rectangle of max/min area inscribed in a circle of radius R
4. the rectangle of max/min perimeter inscribed in a circle of radius R