Find the longest beam that can be carried horizontally around the corner from a hallway of width 3 m to a hallway of width 2 m.

- make a diagram (blackboard): we must find the *minimum* (!) length of a (horizontal) line segment touching 2 walls and the corner – since the beam cannot be longer than this.

- variable: $\theta$ = angle between segment and (2 m) hallway wall
- length: using trig, find length of piece in each hall, and sum:

$$L(\theta) = \frac{3}{\cos(\theta)} + \frac{2}{\sin(\theta)}, \quad 0 < \theta < \frac{\pi}{2}$$

- "endpoints": $\lim_{\theta\to 0+} L(\theta) = +\infty$, $\lim_{\theta\to\frac{\pi}{2}-} L(\theta) = +\infty$
  so the <u>minimum must occur at a critical number</u>
- critical points: $0 = L'(\theta) = \frac{3\sin(\theta)}{\cos^2(\theta)} - \frac{2\cos(\theta)}{\sin^2(\theta)} \implies \tan^3(\theta) = \frac{2}{3}$
- to find $\sin(\theta)$ and $\cos(\theta)$, draw a triangle, or use trig identities;

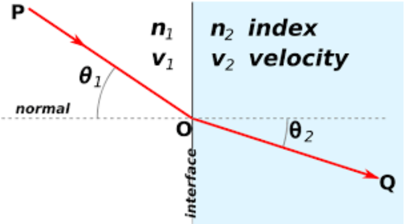
$$\cos(\theta) = \frac{1}{\sqrt{1+\tan^2(\theta)}} = \frac{1}{\sqrt{1+(\frac{2}{3})^{\frac{2}{3}}}}, \quad \sin(\theta) = \frac{\tan(\theta)}{\sqrt{1+\tan^2(\theta)}} = \frac{(\frac{2}{3})^{\frac{1}{3}}}{\sqrt{1+(\frac{2}{3})^{\frac{2}{3}}}}$$

- min. length: $L(\tan^{-1}((\frac{2}{3})^{\frac{1}{3}})) = 3^{\frac{2}{3}}\sqrt{3^{\frac{2}{3}} + 2^{\frac{2}{3}}} + 2^{\frac{2}{3}}\sqrt{3^{\frac{2}{3}} + 2^{\frac{2}{3}}}$

$= \boxed{\left(3^{\frac{2}{3}} + 2^{\frac{2}{3}}\right)^{\frac{3}{2}} \approx 7.02\ m}$

Ponder: what if we use 3 $m$ ceiling?

A light ray crosses an interface between media with different light speeds (equivalently, different *refractive indices*): $v_1 = \frac{c}{n_1}$, $v_2 = \frac{c}{n_2}$



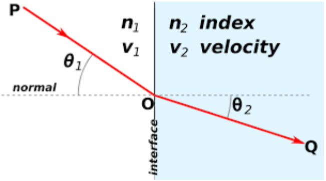

**Fermat's Principle**: ray $P \to Q$ takes path of minimal travel time.

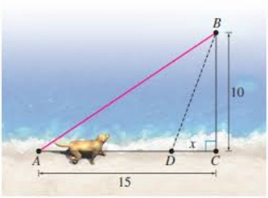
Problem: find refracted angle $\theta_2$ in terms of incident angle $\theta_1$.

- coords: interface $\{x = 0\}$, $P : (x_1, y_1)$, $Q : (x_2, y_2)$, $O : (0, y)$
- travel time: $T(y) = \frac{\sqrt{x_1^2+(y-y_1)^2}}{v_1} + \frac{\sqrt{x_2^2+(y-y_2)^2}}{v_2}$, $-\infty < y < \infty$
- "endpoints": $\lim_{y\to\pm\infty} T(y) = +\infty$, so min. is at a critical #
- critical points:

$$0 = T'(y) = \frac{y-y_1}{v_1\sqrt{x_1^2+(y-y_1)^2}} + \frac{y-y_2}{v_2\sqrt{x_2^2+(y-y_2)^2}} = -\frac{\sin(\theta_1)}{v_1} + \frac{\sin(\theta_2)}{v_2}$$

$$\Longrightarrow \boxed{\frac{\sin(\theta_2)}{\sin(\theta_1)} = \frac{v_2}{v_1} = \frac{n_1}{n_2}}$$ "Snell's Law"




$$\boxed{\frac{\sin(\theta_2)}{\sin(\theta_1)} = \frac{v_2}{v_1} = \frac{n_1}{n_2}}$$

Remember Elvis the dog?

- for Elvis: $v_1 = 10\ m/s,\ v_2 = 2\ m/s,\ \theta_1 = \frac{\pi}{2}$
- so by Snell: $\sin(\theta_2) = \frac{2}{10}\sin(\frac{\pi}{2}) = \frac{1}{5}$

$$\Longrightarrow\ x = 10\tan(\theta_2) = 10\frac{\frac{1}{5}}{\sqrt{1-(\frac{1}{5})^2}} = \frac{10}{\sqrt{24}} \approx 2.04\ m$$

A painting of height $H$ hangs on a wall in front of you, its lower edge a distance $D$ above your eye level. How far back should you stand for optimal viewing?

"Optimal" here means: the angle subtended by the painting from your eye is maximal.

- draw a good diagram to conclude that the subtended angle $\theta$, as a function of your distance $R$ from the wall is

$$\theta(R) = \tan^{-1}\left(\frac{D+H}{R}\right) - \tan^{-1}\left(\frac{D}{R}\right), \quad R > 0$$

- "endpoints": $\lim_{R\to 0+} \theta(R) = \frac{\pi}{2} - \frac{\pi}{2} = 0$, and $\lim_{R\to +\infty} \theta(R) = 0 - 0 = 0$, so max. must occur at a critical #

- critical #s: $0 = \theta'(R) = \frac{1}{1+\left(\frac{D+H}{R}\right)^2}\frac{-(D+H)}{R^2} - \frac{1}{1+\left(\frac{D}{R}\right)^2}\frac{-D}{R^2}$

$$= -\frac{(D+H)}{R^2+(D+H)^2} + \frac{D}{R^2+D^2} \implies (D+H)(R^2+D^2) = D(R^2+(D+H)^2)$$
$$\implies HR^2 = D(D+H)^2 - (D+H)D^2 = D(D+H)H$$
$$\implies R^2 = D(D+H),$$

and the 'optimal' viewing distance is $\boxed{R = \sqrt{D(D+H)}}$

Some more fun optimization problems...

1. a projectile is launched at some angle from the horizon, experiencing negligible drag. Find the launch angle which maximizes the (horizontal) distance traveled
   - on level ground
   - down a (infinite, flat) hill at angle $\alpha$ from the horizon

2. find the line through the origin which "best fits" a set of data points $(x_1, y_1)$, $(x_2, y_2), \ldots, (x_N, y_N)$

3. find the isoceles triangle minimizing the ratio $\frac{(\text{perimeter})^2}{\text{area}}$