Yet more ODE: quick review

Consider the ODE IVP

$$\frac{dP}{dt} = P(2 - P), \quad P(0) = P_0$$

1. what words describe this ODE?
   – first-order, separable, autonomous, "logistic"
2. What are the steady states (equilibria)? $P = 0, \quad P = 2$
3. Sketch the solutions for a few choices of P_0.
4. Which steady states are stable? Unstable?
5. How might you solve this ODE explicitly? $\int \frac{dP}{P(2-P)} = \int dt$
6. Verify that $P(t) = \frac{2}{1+Ce^{-2t}}$ solves this problem, for some C.
   $P' = \frac{4Ce^{-2t}}{(1+Ce^{-2t})^2} \qquad 2P - P^2 = \frac{4(1+Ce^{-2t})-4}{(1+Ce^{-2t})^2} = \frac{4Ce^{-2t}}{(1+Ce^{-2t})^2}$
   so the ODE is satisfied. The initial condition requires
   $P_0 = P(0) = \frac{2}{1+C} \implies C = \frac{2}{P_0} - 1.$

Yet more ODE: Euler's Numerical Method (review)

Confronted with a first-order ODE IVP

$$y'(t) = F(y(t), t), \qquad y(t_0) = y_0$$

that we cannot solve explicitly (which is almost all of them!), the simplest **numerical method** for generating an approximate solution is **Euler's method**: given a **step size** Δt, we use the linear approximation for $y(t_0 + \Delta t)$ at $t = t_0$:

$$y(t_0 + \Delta t) \approx y(t_0) + y'(t_0)\Delta t = y_0 + F(y_0, t_0)\Delta t =: y_1,$$

for the next time step, use linear approximation at $t_1 := t_0 + \Delta t$:

$$y(t_0+2\Delta t) = y(t_1+\Delta t) \approx y(t_1)+y'(t_1)\Delta t \approx y_1+F(y_1, t_1)\Delta t =: y_2,$$

and continue this way (for as long as you require)

Euler's Method: $\qquad t_n = t_0 + n\Delta t,$

$$y(t_{n+1}) \approx y_{n+1} = y_n + F(y_n, t_n)\Delta t$$

Yet more ODE: Euler's Numerical Method (review)

Example: Compute the first 3 steps in the Euler method, using step size 1/2, for the logistic IVP: $\frac{dP}{dt} = P(2-P), \quad P(0) = 1.$
Compare with exact solution $P(t) = \frac{2}{1+e^{-2t}}$. Repeat for $\Delta t = 1/6$.

$P_0 = 1, \qquad P_1 = P_0 + P_0(2-P_0)\Delta t = 1 + \Delta t,$
$P_2 = P_1 + P_1(2-P_1)\Delta t = 1 + \Delta t + (1+\Delta t)(1-\Delta t)\Delta t$
$\quad = 1 + 2\Delta t - (\Delta t)^3, \quad$ etc...

	$\Delta t = 1/2 = 0.5$			$\Delta t = 1/6 \approx 0.17$		
n	t_n	P_n	error	t_n	P_n	error
0	0.0	1.000	0.000	0.00	1.000	0.000
1	0.5	1.500	0.038	0.17	1.167	0.002
2	1.0	1.875	0.113	0.33	1.329	0.007
3	1.5	1.992	0.087	0.50	1.477	0.015

- error in one step (linear approximation) is (roughly) $\propto (\Delta t)^2$
- so cumulative error solving up to $t = T = N\Delta t$ is $\propto (\Delta t)$

(Euler is a "first-order" numerical method)

Example of ODE system: a simple predator-prey model

- populations: $P(t) =$ predators, $N(t) =$ prey
- without predators, prey population grows: $\frac{dN}{dt} = rN$
- without prey, predator population decays: $\frac{dP}{dt} = -kP$
- rate of predation: proportional to NP

$$\left\{ \begin{aligned} \frac{dN}{dt} &= rN - pNP \\ \frac{dP}{dt} &= -kP + \alpha pNP \end{aligned} \right\} \qquad P(0) = P_0, \quad N(0) = N_0$$

"Lotka-Volterra" model

What are the equilibria (constant) solutions of this **system**?

$N(r - pP) = 0$ and $P(-k + \alpha pN) = 0$

$\implies (P, N) = (0, 0)$ or $(P, N) = (\frac{r}{p}, \frac{k}{\alpha p})$

Set up Euler's method for the Lotka-Volterra IVP:

$$n = 0, 1, 2, \dots, \qquad t_n = n\Delta t$$
$$P(t_{n+1}) \approx P_{n+1} = P_n + (rN_n - pN_nP_n)\Delta t$$
$$N(t_{n+1}) \approx N_{n+1} = N_n + (-kP_n + \alpha pN_nP_n)\Delta t$$

“Phase portrait” of the Lotka-Volterra system

Can you interpret this plot?