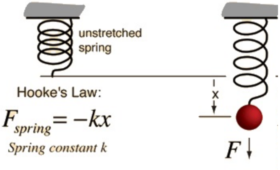


Why is Hooke's Law so useful?

# LINEAR APPROXIMATION

If $f(x)$ is differentiable at $x = a$,

The **linearization** of $f$ at $a$ is the linear function

$$L(x) = f(a) + f'(a)(x - a)$$

and the approximation

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

is the **linear** (or **tangent line**) **approximation to $f$ near $a$.**

*Example*: Use the linear approximation for $x^{1/3}$ at $x = 27$ to find an approximate value for $(27.5)^{1/3}$:

$$f(x) = x^{1/3}, \quad f(27) = 3$$

$$f'(x) = \tfrac{1}{3}x^{-2/3}, \quad f'(27) = \tfrac{1}{3}(27)^{-2/3} = \tfrac{1}{27},$$

$$f(27.5) \approx f(27) + f'(27)(0.5) = 3 + \tfrac{1}{54} = \tfrac{163}{54}$$

(calculator: $\frac{163}{54} \approx 3.0185, \quad (27.5)^{1/3} \approx 3.0184$)

# LINEAR APPROXIMATION

Linear approximation re-written: if $y = f(x)$,

$$x \mapsto x + \Delta x \implies y \mapsto y + \Delta y$$

$$\Delta y = f(x + \Delta x) - f(x) \approx f'(x)\Delta x = \frac{dy}{dx}\Delta x.$$

*Example*: an object of mass $m$ is moving along a line.
The force on it is a (known) function $F(v)$ of its velocity.
You know the velocity $v$ at a particular time $t$.
Use linear approximation to estimate $v$ a (short) time $\Delta t$ later:

Newton: $\frac{dv}{dt} = \frac{1}{m}F(v)$
linear approx: $\Delta v = v(t + \Delta t) - v(t) \approx \frac{dv}{dt}\Delta t = \frac{1}{m}F(v(t))\Delta t$
so: $v(t + \Delta t) \approx v(t) + \frac{1}{m}F(v(t))\Delta t$

Now estimate $v(t + 2\Delta t)$:

# LINEAR APPROXIMATION

Now estimate $v(t+2\Delta t)$:

*One way*: $v(t+2\Delta t) \approx v(t) + \frac{1}{m}F(v(t))(2\Delta t)$

*Another way*: use our previous approximation of $v(t+\Delta t)$ to do a linear approximation at $(t+\Delta t, v(t+\Delta t))$:

$$\begin{aligned} v(t+2\Delta t) &\approx v(t+\Delta t) + \frac{1}{m}F(v(t+\Delta t))\Delta t \\ &\approx v(t) + \frac{1}{m}F(v(t))\Delta t + \frac{1}{m}F\left(v(t) + \frac{1}{m}F(v(t))\Delta t\right)\Delta t \end{aligned}$$

(iterating this procedure is called.... **Euler's Method** !)

# LINEAR APPROXIMATION

How much error we are making by using the linear approximation?

*Theorem*: Suppose $f''(x)$ exists for all $x$ near $a$. Then for any $x$ near $a$ there exists a point $s$ between $a$ and $x$ such that

$$f(x) - L(x) = f(x) - f(a) - f'(a)(x-a) = \frac{1}{2} f''(s)(x-a)^2.$$

Takeaway: for increment $\Delta x$, **the error is roughly of size** $(\Delta x)^2$.

*Question*: given initial velocity $v(0)$, and Newton's equation $v'(t) = \frac{1}{m} F(v(t))$, you estimate $v(1)$ using Euler's method with step-size $\Delta t$. What is the rough size of your error?

– error per step is roughly $(\Delta t)^2$
– # of steps is $1/\Delta t$
– so total error is roughly $\Delta t$

Approximately what volume of paint is required to cover a sphere of 100 m radius to a 1 mm thickness?

Our first instinct might be to say: easy! – since the surface area of the sphere is $4\pi(100)^2$ m, $V_{paint} \approx 4\pi(100)^2 \left(\frac{1}{1000}\right) = 40\pi$ m$^3$.

This is indeed a good approximation, and it is coming from linear approximation: since the volume inside a sphere of radius $R$ is $V(R) = \frac{4}{3}\pi R^3$, and $\frac{dV}{dR} = 4\pi R^2$, linear approximation, with $R = 100$ and $\Delta R = \frac{1}{1000}$, says

$$\begin{aligned} V_{paint} &= V(R + \Delta R) - V(R) \approx V'(R)\Delta R = 4\pi R^2 \Delta R \\ &= 4\pi(100)^2 \frac{1}{1000} = 40\pi \ m^3. \end{aligned}$$