IMPLICIT DIFFERENTIATION

Example: Find the slope of the tangent line to each point of the circle of radius R centred at a point (a, b) in the xy-plane:

• equation of this circle: $(x - a)^2 + (y - b)^2 = R^2$

• one approach: solve to get y as a function of x, then differentiate to find slope – any problem with this?

• another approach: take the view that the equation **implicitly** (i.e. not explicitly) defines y as function of x, and differentiate both sides, remembering the chain rule:

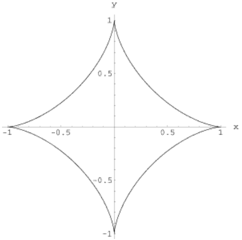
$$2(x - a) + 2(y - b)\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{x - a}{y - b} .$$

This is called **implicit differentiation**

IMPLICIT DIFFERENTIATION

Example: Find the slope of the tangent line to each point of the

"astroid" $\quad x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$



$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{\frac{2}{3}x^{-\frac{1}{3}}}{\frac{2}{3}y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

What is happening at the "corners"?

IMPLICIT DIFFERENTIATION

Example: if $x^3 + 3y + y^3 - y^2 = 4$, find

- y' (in terms of x and y)
- y'' when $x = 1$

IMPLICIT DIFFERENTIATION

Example: (inverse trig. function): $\sin^{-1}(x) = \arcsin(x)$ is the inverse function of $\sin(x)$ restricted to $-\pi/2 \le x \le \pi/2$:

$$y = \sin^{-1}(x) \;\Leftrightarrow\; x = \sin(y) \text{ and } -\pi/2 \le y \le \pi/2$$

Sketch the graph of $\arcsin(x)$ and find $\frac{d}{dx}\arcsin(x)$:

- implicit diff: $x = \sin(y) \implies 1 = \cos(y)\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1}{\cos(y)}$
- express $\cos(y) = \pm\sqrt{1-\sin^2(y)} = +\sqrt{1-\sin^2(y)} = \sqrt{1-x^2}$
- so: $\boxed{\frac{d}{dx} sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \quad (-1 < x < 1)}$

Example: do the same for $\tan^{-1}(x) = \arctan(x)$

- $y = \tan^{-1}(x) \;\Leftrightarrow\; x = \tan(y)$ and $-\pi/2 \le y \le \pi/2$
- $1 = \sec^2(y)\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)}$
- so: $\boxed{\frac{d}{dx} tan^{-1}(x) = \frac{1}{1+x^2}}$

LOGARITHMS

$\ln(x)$ is the inverse to e^x: $\ln(e^y) = y$, $e^{\ln(x)} = x$ $(x > 0)$

- $e^0 = 1, \ e^1 = e$
- $e^{y+w} = e^y e^w$
- $e^{-y} = \frac{1}{e^y}$
- $e^{y-w} = \frac{e^y}{e^w}$
- $(e^y)^r = e^{ry}$

- $\ln(1) = 0, \ \ln(e) = 1$
- $\ln(xz) = \ln(x) + \ln(z)$
- $\ln(\frac{1}{x}) = -\ln(x)$
- $\ln(\frac{x}{z}) = \ln(x) - \ln(z)$
- $\ln(x^r) = r\ln(x)$

- derivatives: $\boxed{\frac{d}{dy} e^y = e^y}$

$$y = ln(x) \implies x = e^y \implies \text{(implicit diff.) } 1 = e^y \frac{dy}{dx}$$

$$\implies \frac{dy}{dx} = \frac{1}{e^y} \implies \boxed{\frac{d}{dx} \ln(x) = \frac{1}{x} \ \ (x > 0)}$$

- general exp'ls: for $a > 0$ $\boxed{a^x = e^{x\ln(a)}}$ (as $\ln(a^x) = x\ln(a)$)

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x\ln(a)} = \ln(a) e^{x\ln(a)} = \ln(a) a^x$$

- general logs: for $a > 0$, $a \neq 1$, $\log_a(x)$ is inverse to a^x:

$$a^{\log_a(x)} = x \implies \log_a(x)\ln(a) = \ln(x) \implies \boxed{\log_a(x) = \frac{\ln(x)}{\ln(a)}}$$

LOGARITHMS

Example: using **'logarithmic differentiation'**, find $\frac{d}{dx}$ of:

$$3^{x^2} \qquad x^x \qquad (x-1)(x-2)(x-3)\cdots(x-n) \qquad x^{x^x}$$

- $y = 3^{x^2} \implies \ln(y) = x^2\ln(3) \implies \frac{1}{y}\frac{dy}{dx} = 2x\ln(3)$

$$\implies \frac{dy}{dx} = 2\ln(3)xy = \boxed{2\ln(3)x3^{x^2}}$$

- $y = x^x \implies \ln(y) = x\ln(x) \implies \frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \ln(x) = 1 + \ln(x)$

$$\implies \frac{dy}{dx} = (1+\ln(x))y = \boxed{(1+\ln(x))x^x}$$

- $y = (x-1)(x-2)(x-3)\cdots(x-n)$

$$\implies \ln(y) = \ln(x-1) + \ln(x-2) + \cdots + \ln(x-n)$$

$$\implies \frac{1}{y}\frac{dy}{dx} = \frac{1}{x-1} + \frac{1}{x-2} + \cdots + \frac{1}{x-n} \quad \implies \quad \frac{dy}{dx} =$$

$$\boxed{\left(\frac{1}{x-1} + \frac{1}{x-2} + \cdots + \frac{1}{x-n}\right)(x-1)(x-2)(x-3)\cdots(x-n)}$$