7 marks 1. Evaluate the following integrals.

(a) $\int \frac{4}{y^2-4}\,dy = \int \frac{4}{(y-2)(y+2)}\,dy = \int \frac{dy}{y-2} + \int \frac{-dy}{y+2} =$

$\frac{A}{y-2} + \frac{B}{y+2} = \frac{4}{(y-2)(y+2)}$ $\quad = \ln|y-2| - \ln|y+2| + C$

$A=1$, $B=-1$

(b) $\int_0^2 \frac{x}{(3x^2+5)^n}\,dx = \frac{1}{6}\int_5^{17} \frac{du}{u^n}$

$u = 3x^2+5$ $\quad x=0,\ u=5$

$du = 6x\,dx$ $\quad x=2,\ u=17$

if $n \neq 1$ then

$= \frac{1}{6}\frac{u^{-n+1}}{-n+1}\Big|_5^{17} = \frac{1}{6(1-n)}(17^{1-n} - 5^{1-n})$

if $n=1$ then

$= \frac{1}{6}\ln(u)\Big|_5^{17} = \frac{1}{6}(\ln 17 - \ln 5)$

(c) $\int_1^e z^3 \ln z\,dz$

$u = \ln z$, $dv = z^3$

$du = \frac{1}{z}dz$ $\quad v = \frac{z^4}{4}$

$= \frac{1}{4}z^4 \ln z\Big|_1^e - \int_1^e \frac{1}{4}z^4 \cdot \frac{1}{z}\,dz =$

$= \frac{1}{4}e^4 - \int_1^e \frac{1}{4}z^3\,dz = \frac{1}{4}e^4 - \frac{1}{4}\frac{z^4}{4}\Big|_1^e = \frac{1}{4}e^4 - \frac{e^4}{16} + \frac{1}{16}$

$= \frac{3e^4+1}{16}$

12 marks 2. Evaluate any **three** integrals of your choice from the list below. Remember to include integration constants whenever appropriate. Continue your work on the next page if you need more space.

(a) $\int \frac{x^3}{\sqrt{4-x^2}}\,dx$ (b) $\int x^3 e^{-x^2}\,dx$

(c) $\int \frac{2t^2}{t^3-3t^2+2t}\,dt$ (d) $\int_0^1 \frac{x}{x^2+x+1}\,dx$

(a) $x = 2\sin\theta$, $dx = 2\cos\theta\,d\theta$

$\int \frac{x^3\,dx}{2\sqrt{1-(\frac{x}{2})^2}} = \int \frac{8\sin^3\theta \cdot \cancel{2}\cancel{\cos\theta}\,d\theta}{\cancel{2}\cancel{\cos\theta}} = 8\int \sin^3\theta\,d\theta = 8\int \sin^2\theta \cdot \sin\theta\,d\theta$

$= 8\int (1-\cos^2\theta)\sin\theta\,d\theta = 8\int (u^2-1)\,du = 8\left(\frac{u^3}{3} - u\right) + C = \frac{8}{3}\cos^3\theta - 8\cos\theta + C$

$u = \cos\theta$, $du = -\sin\theta\,d\theta$

$x = 2\sin\theta \rightarrow \cos\theta = \sqrt{1-(x/2)^2}$

$= \frac{8}{3}\left(1-\frac{x^2}{4}\right)^{3/2} - 8\sqrt{1-\frac{x^2}{4}} + C$

(b) $\int x^3 \cdot e^{-x^2}\,dx = \frac{1}{2}\int u e^{-u}\,du = -\frac{1}{2}ue^{-u} - \frac{1}{2}\int -e^{-u}\,du =$

$u = x^2$, $du = 2x\,dx$ $\quad U = u$, $dv = e^{-u}\,du$ $\quad dU = du$, $v = -e^{-u}$

$= -\frac{1}{2}ue^{-u} - \frac{1}{2}e^{-u} + C$

$= -\frac{1}{2}e^{-x^2}(x^2+1) + C$

(c) $\int \frac{2t^{\cancel{2}}}{\cancel{t}(t-2)(t-1)}\,dt = \int \frac{4}{t-2}\,dt - \int \frac{2}{t-1}\,dt =$

$\frac{A}{t-2} + \frac{B}{t-1} = \frac{2t}{(t-2)(t-1)}$

$= 4\ln|t-2| - 2\ln|t-1| + C$

$A = 4$, $B = -2$

This page has been left blank for your workings and solutions.

(d) $$\int_0^1 \frac{x\,dx}{x^2+x+1} = \frac{1}{2}\int_0^1 \frac{2x+1-1}{x^2+x+1}\,dx = \frac{1}{2}\int_0^1 \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int_0^1 \frac{dx}{x^2+x+1}$$

$u = x^2+x+1$

$du = (2x+1)dx$

complete the square

$x^2+x+1 = (x+\frac{1}{2})^2+\frac{3}{4}$

$$\frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|x^2+x+1| + C$$

$$\frac{1}{2}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \frac{1}{2}\int \frac{4}{3}\,\frac{dx}{\left(\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right)^2+1} =$$

$$= \frac{1}{2}\cdot\frac{4}{3}\int \frac{dx}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1} = \frac{2}{3}\int \frac{\frac{\sqrt{3}}{2}\,du}{u^2+1} = \frac{\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

$\frac{2x+1}{\sqrt{3}} = u$

$\frac{2}{\sqrt{3}}dx = du$

so $$\int_0^1 \frac{x\,dx}{x^2+x+1} = \frac{1}{2}\ln|x^2+x+1| - \frac{\sqrt{3}}{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)\Bigg|_0^1 =$$

$$= \frac{1}{2}\ln(3) - \frac{\sqrt{3}}{3}\arctan\left(\frac{3}{\sqrt{3}}\right) + \frac{\sqrt{3}}{3}\arctan\left(\frac{1}{\sqrt{3}}\right)$$

4 marks 3. Determine whether each of the following integrals is convergent or divergent, and if convergent find its value. Make sure you justify your claims.

(a) $\int_2^\infty \frac{1}{x\ln(x)}dx = \int_{\ln 2}^{\infty} \frac{du}{u} = \lim_{u\to\infty} \ln u - \ln(\ln 2) = \infty$

$u = \ln x$, $x=2 \Rightarrow u = \ln(2)$

$du = \frac{1}{x}dx$, $x\to\infty \Rightarrow u\to\infty$

so integral diverges

(b) $\int_0^3 \frac{dx}{(x-1)^{1/3}} = \int_0^1 \frac{dx}{(x-1)^{1/3}} + \int_1^3 \frac{dx}{(x-1)^{1/3}}$

$u = x-1$, $x=0 \Rightarrow u=-1$

$du = dx$, $x=3 \Rightarrow u=2$

$x=1 \Rightarrow u=0$

$= \int_{-1}^{0} u^{-1/3}du + \int_0^2 u^{-1/3}du =$

$= \lim_{u\to 0^-}\left(\frac{3}{2}u^{2/3} - \frac{3}{2}(-1)^{2/3}\right) + \lim_{u\to 0^+}\left(\frac{3}{2}(2)^{2/3} - \frac{3}{2}u^{2/3}\right)$

$= -\frac{3}{2} + \frac{3}{2}2^{2/3} = \frac{3}{2}\left(\sqrt[3]{4}-1\right)$

6 marks 4. Determine (with justification) whether each of the improper integrals converges or diverges

(a) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x+x^3}}\,dx. = \int_0^1 \frac{dx}{\sqrt{x+x^3}} + \int_1^\infty \frac{dx}{\sqrt{x+x^3}}$

for $0 \le x \le 1$ $\quad 0 \le \dfrac{1}{\sqrt{x+x^3}} \le \dfrac{1}{\sqrt{x}}$ (because $\sqrt{x+x^3} > \sqrt{x}$)

since $\int_0^1 \frac{1}{x^{1/2}}\,dx$ converges, so does $\int_0^1 \frac{1}{\sqrt{x+x^3}}\,dx$

for $x \ge 1$ $\quad \dfrac{1}{\sqrt{x+x^3}} \le \dfrac{1}{\sqrt{x^3}}$ (because $\sqrt{x+x^3} > \sqrt{x^3}$)

since $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ converges, so does $\int_1^\infty \frac{1}{\sqrt{x+x^3}}\,dx$

So the original integral converges.

(b) $\displaystyle\int_0^1 \frac{e^x}{x}\,dx.$

for $0 \le x \le 1$ $\quad \dfrac{e^x}{x} \ge \dfrac{1}{x}$

$\int_0^1 \frac{1}{x}\,dx$ diverges, so $\int_0^1 \frac{e^x}{x}\,dx$ diverges

by comparison