Science One Math

November 12, 2018

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- If $f'(x) = 0$ for all x in interval I $\Rightarrow$ f is constant on I
- If $f'(x) = g'(x) \Rightarrow f(x) = g(x) + C$ for any constant C

A rigorous proof of each of the statements above relies on one single theorem:

the **Mean Value Theorem (MVT)**

(generalized version of Rolle's theorem)

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Today's Goals:

- state the MVT (and Rolle's theorem)
- discuss the importance of hypotheses and meaning of conclusions of theorems
- draw a sketch to illustrate the conclusion of these theorems
- Use the theorem to prove simple theoretical facts or applications

Rolle's theorem *(proved in 1691)*

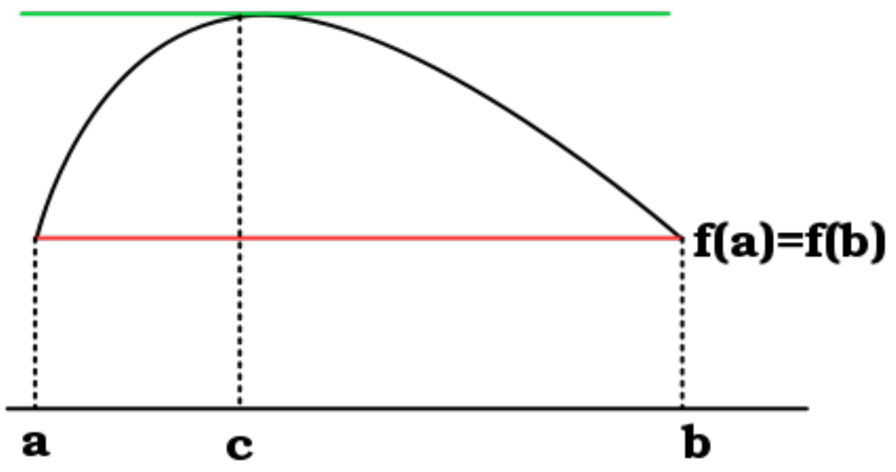
If f is *continuous* on $[a, b]$

differentiable on (a, b)

$f(a) = f(b)$

Then there is a number c in (a, b)

such that $f'(c) = 0$.



Theoretical application: If $P(x)$ is a polynomial, then between any two roots of P there is a root of $P'(x)$.

Physical application: throw an object straight up and then catch it, at some time during its flight it must be stationary

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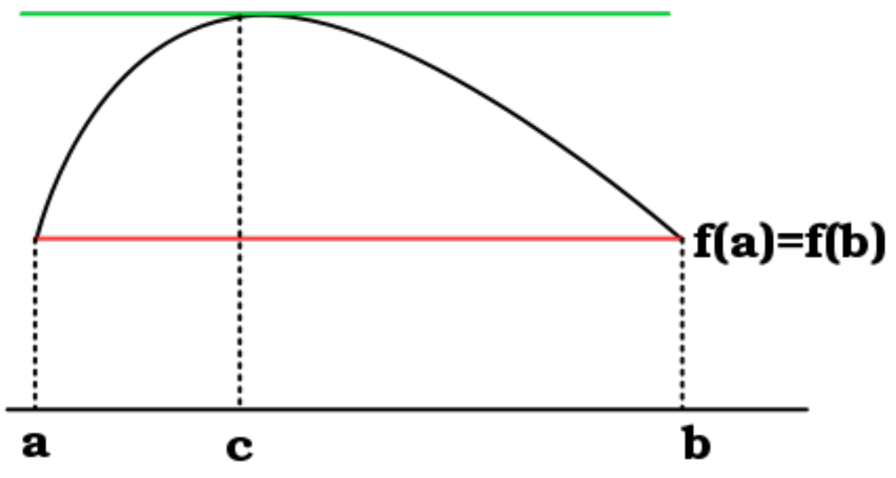
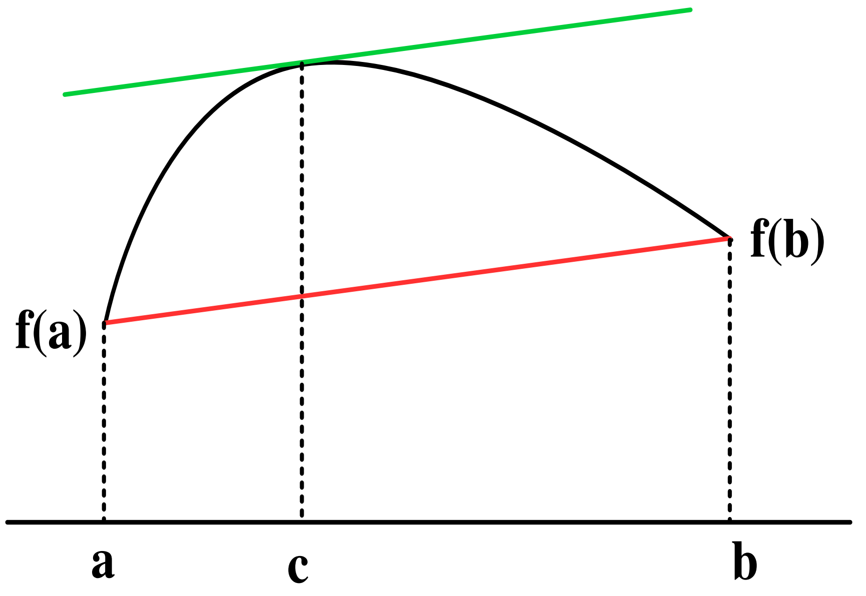


can be generalized to case $f(a) \neq f(b)$.

$\Rightarrow$ the Mean Value Theorem



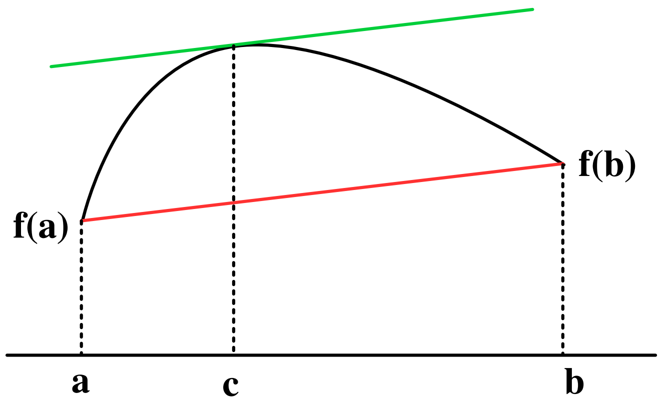
Mean Value Theorem *(proved by Cauchy in 1823)*

If f is *continuous* on $[a, b]$

differentiable on (a, b)

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Mean Value Theorem

If f is *continuous* on $[a, b]$

differentiable on (a, b)

Then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Proof: Construct a new function $\mathrm{G}(x) = f(x) - (y_{secant})$

$$\mathrm{G}(x) = f(x) - [f(a) + \frac{f(b)-f(a)}{b-a}(x-a)]$$

Note $\mathrm{G}(x)$ satisfies assumptions of Rolle's theorem on $[a, b]$:

continuous on $[a, b]$, differentiable on (a, b), and $G(a) = G(b)$.

Thus by Rolle's thrm, there exists c in (a, b) such that $G'(c) = 0$. This yields

$$G'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \quad \Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Mean Value Theorem

If f is *continuous* on $[a, b]$

differentiable on (a, b)

Then there is a number c in (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Why do we need both assumptions?

Draw examples of functions that fail to satisfy

- continuity on $[a, b]$
- differentiability on (a, b)

Applications of MVT : a few examples…

Example 1.1

On a toll road a driver takes a time stamped toll-card from the starting booth and drives directly to the end of the toll section. After paying the required toll, the driver is surprised to receive a speeding ticket along with the toll receipt. The driver complains, but the booth attendant who aced her calculus exam showed that the driver was speeding at least once during his trip. How did she do that?

Example 1.2

Two racers start a race at the same moment and finish in a tie. Which of the following must be true?

A. At some point during the race the two races were not tied.
B. The racers’ speeds at the end of the race must have been exactly the same.
C. The racers must have had the same speed at exactly the same time at some point in the race.
D. The racers had to have the same speed at some moment, but not necessarily at exactly the same time.

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Example 2

Suppose $f(0) = -3$ and $f\,'(x) \leq 5$ for all x. How large can $f(3)$ possibly be?

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Solution: By MVT on [0, 3], there is a number c such that

$$f'(c) = \frac{f(3) - f(0)}{3 - 0} \leq 5$$

$$f(3) + 3 \leq 15$$

$$f(3) \leq 12.$$

Example 3

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Proof: Let $f(x) = x^3 + 5x - 18$. Observe $f(1) < 0$ and $f(3) > 0$. Since f is continuous everywhere (it's a poly.), by IVT there must be a number c such that $f(c) = 0$. In this case it is easy to see that $c = 2$.

Now we prove (by contradiction) that the equation above has only one root.

Suppose there is a second root $b > c$, i.e. $f(b) = 0$. Since f is differentiable everywhere (it's a poly), it satisfied Rolles' theorem on $[c, b]$. By Rolle's theorem, there must be a number $c < d < b$ such that $f'(d) = 0$. However, $f'(x) = 3x^2 + 5 > 0$ for all x (i.e. $f'(x) \neq 0$). Therefore, our initial assumption about a second root must be false. We proved there is only one solution.

Example 4.1

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Proof: Consider $a < b$ for any a and b in I. By MVT there is a number c in $[a, b]$ such that

$$x = 0 \Longrightarrow f(b) - f(a) = 0 \Longrightarrow f(b) = f(a)$$

Similarly if $a > b$. Thus we proved that the function takes the same value at any point in $\mathcal{I}$, that is, f is constant on $\mathcal{I}$.

Example 4.2

If $f'(x) = g'(x)$ for all x in an interval $\mathcal{I}$, then $f(x) = g(x) + C$ for some constant *C*.

This is true because

A. after differentiating $f(x) = g(x) + C$ implicitly, we get $f'(x) = g'(x)$.

B. the function $h(x) = f(x) - g(x)$ has zero derivative in $\mathcal{I}$, thus $h(x) = C$ for all x in $\mathcal{I}$, where C is a constant.

C. the derivative of a constant is zero.

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C. the derivative of a constant is zero.

MVT: a cornerstone of Calculus

local view $\Longrightarrow$ global view

(value of the derivative at point inside an interval) (behaviour of *f* over an interval, i.e. increasing/decreasing)