Science One Math

November 20, 2018

Shape of a curve

Last time:

- candidates for local extrema are **critical points**.
- a critical point of f is not necessarily an extreme value of f.
- need a method for testing critical points.

Last time: Good candidates for extrema are points where $f'=0$ and f' DNE.

Defn: A **critical number** of f is a number c (in the domain of f) such that either $f'(c)=0$ or $f'(c)$ is undefined.

Does f always attain a local extremum at a critical number?

Last time: Good candidates for extrema are points where $f'=0$ and f' DNE.

Defn: A **critical number** of f is a number c (in the domain of f) such that either $f'(c)=0$ or $f'(c)$ is undefined.

Does f always attain a local extremum at a critical number? Not always!

e.g. $f(x)=x^3$ no extremum at $x=0$ even though $f'(0)=0$.

We need a test to identify which critical numbers correspond to local extrema of f.

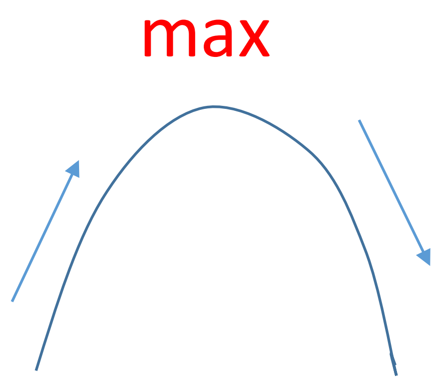
☞ first derivative test

☞ second derivative test

How to locate local/relative extrema

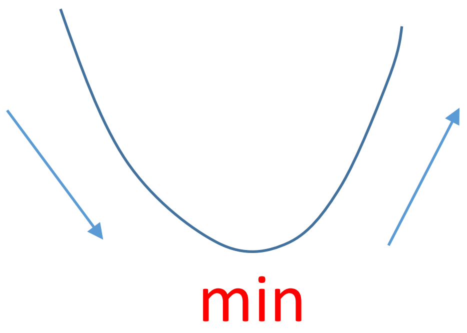
Observation: If $f(c)$ is a local maximum, then

$f(x)$ is **increasing** for $x < c$ and **decreasing** for $x > c$.



If $f(c)$ is a local minimum, then

$f(x)$ is **decreasing** for $x < c$ and **increasing** for $x > c$.



First derivative test for extrema

If $x = c$ is a **critical number** of f and:

$f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ $\Rightarrow f(c)$ is a **maximum**

$f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$ $\Rightarrow f(c)$ is a **minimum**

Suppose that $g(x)$ is a function undefined at $x = 2$ and continuous for all $x \neq 2$ whose derivative is

$$g'(x) = \frac{(x+4)(x-1)^2}{x-2}.$$

A. g has a local maximum at $x = -4$ and a local minimum at $x = 2$.

B. g has a local maximum at $x = -4$ and no local minima.

C. g is increasing for all $x \neq 2$.

D. g has a local minimum at $x = 2$ and no local maxima.

Second derivative test for extrema

Observation 1: "....when $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$..."

means "...when f' decreases as x increases through c..."

Observation 2: If we apply the theorem "If $f'(x) > 0$ for all x in $\mathcal{I}$, f is increasing on $\mathcal{I}$" to the first derivative function, we have

" if $(f')'(x) > 0$ for all x in $\mathcal{I}$ $\Rightarrow$ f' is increasing on $\mathcal{I}$ "

$\Rightarrow$ Observation 1 and 2 lead to the second-derivative test for extrema

2nd-derivative test for extrema: If $x = c$ is a critical number of f and

If $f''(c) < 0$ on an interval containing c $\Rightarrow f(c)$ is a **maximum.**

If $f''(c) > 0$ on an interval containing c $\Rightarrow f(c)$ is a **minimum.**

More about the shape of a curve: Concavity

Observation 2: If $f''(x) > 0$ for all x in $\mathcal{I}$ $\Rightarrow$ f' is increasing on $\mathcal{I}$.

What does this information tell us about the shape of f ?

$\Rightarrow$ the slope of tangent is increasing, the graph of f bends upwards

$\Rightarrow$ the graph of f is **above tangent line** at any point in $\mathcal{I}$

Proof : For any $a < x < b$, by MVT there is a number c in (a, x) such that

$\frac{f(x)-f(a)}{x-a} = f'(c) > f'(a)$ because f' is increasing, it follows

$$f(x) - f(a) > f'(a)(x-a)$$

eq. of tangent at $x = a$

$$f(x) > f(a) + f'(a)(x-a)$$

Defn f is **concave up** on $\mathcal{I}$ if f' is increasing on $\mathcal{I}$ (above tangent)

f is **concave down** on $\mathcal{I}$ if f' is decreasing on $\mathcal{I}$ (below tangent)

Concavity Test

If $f''(x) > 0$ for all x in $\mathcal{I}$ $\Rightarrow$ f is concave up on $\mathcal{I}$.

If $f''(x) < 0$ for all x in $\mathcal{I}$ $\Rightarrow$ f is concave down on $\mathcal{I}$.

Defn If the concavity changes as x goes through c, $f(c)$ is called **inflection point.**

Example: Find the concavity and all inflection points of $f(x) = x^{\frac{1}{3}}(x+4)$.

Example: Find the concavity and all inflection points of $f(x) = x^{\frac{1}{3}}(x+4)$.

$f'(x) = \frac{4(x+1)}{3x^{2/3}}$ and $f''(x) = \frac{4x-8}{9x^{5/3}}$.

$f''(x) = 0$ at $x = 2$ and $f''(x)$ DNE at $x = 0$.

x	$(-\infty, 0)$	0	$(0, 2)$	2	$(2, \infty)$
Sign of f''	$+$	DNE	$-$	0	$+$
Behaviour of f	CONC. UP	I.P.	CONC. DOWN	I.P.	CONC. UP

Sketch a graph of $y = f(x)$.

Example: Find the concavity and all inflection points of $f(x) = x^{\frac{1}{3}}(x+4)$.

$f'(x) = \frac{4(x+1)}{3x^{2/3}}$ and $f''(x) = \frac{4x-8}{9x^{5/3}}$.

$f''(x) = 0$ at $x = 2$ and $f''(x)$ DNE at $x = 0$.

x	$(-\infty, 0)$	0	$(0, 2)$	2	$(2, \infty)$
Sign of f''	+	DNE	−	0	+
Behaviour of f	CONC. UP	I.P.	CONC. DOWN	I.P.	CONC. UP

Sketch a graph of $y = f(x)$.

x	$(-\infty, -1)$	-1	$(-1, 0)$	0	$(0, +\infty)$
Sign of f'	−	0	+	DNE	+
Behaviour of f	DEC	MIN	INC	Vertical tangent	INC

Find all local extrema and discuss concavity of the following functions

$$y = x \ln x$$

$$y = \frac{x^2}{x^2 - a^2}$$ where a is a constant

$$y = x^2 e^x$$