Science One Math

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Shape of a curve

What kind of information about f can we can extract from the derivatives?

- the **first derivative** tells us about (rate of) **change in function values** (do they increase or decrease when x increases?)
- the **second derivative** tells us about (rate of) **change in slope** (does the slope increase or decrease?)
- derivatives allow us to locate local extreme values of a function
  - First-derivative test
  - Second-derivative test

Summary of tests

Increasing/Decreasing test

If $f'(x) > 0$ for all x in $\mathcal{I}$, f is increasing on $\mathcal{I}$.

If $f'(x) < 0$ for all x in $\mathcal{I}$, f is decreasing on $\mathcal{I}$.

First derivative test for extrema

If $x = c$ is a **critical number** of f and:

- $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$ $\Rightarrow f(c)$ is a **maximum**
- $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$ $\Rightarrow f(c)$ is a **minimum**

Second derivative test for extrema

If $x = c$ is a **critical number** of f and $f''(c) < 0 \Rightarrow f(c)$ is a **maximum.**

If $x = c$ is a **critical number** of f and $f''(c) > 0 \Rightarrow f(c)$ is a **minimum.**

Let’s test these ideas….

If $f(p)$ is not a local extremum of f, then $x = p$ is not a critical point of f.

A. True, I’m confident.
B. True, I’m not confident.
C. False, I’m confident.
D. False, I’m not confident.

Let f be a twice differentiable function everywhere, then f' is maximum at a point of inflection of f.

A. True, I'm confident.
B. True, I'm not confident.
C. False, I'm confident.
D. False, I'm not confident.

Consider two functions f and g, both differentiable for all x. Suppose $f(x)$ is increasing for all x and $g(x)$ is decreasing for all x.

Then $f(g(x))$ is also increasing for all x.

A. True, I'm confident.
B. True, I'm not confident.
C. False, I'm confident.
D. False, I'm not confident.

Shape of curve

Useful information to sketch the graph of a function

- Domain
- Interval of increase/decrease
- Concavity
- Local extrema
- Inflection points
- Other useful points: intercepts, points at which $f(x) \to \infty$ or $f(x) \to -\infty$ (vertical asymptotes)
- End behaviour of $f(x)$ as $x \to +\infty$ and $x \to -\infty$ (horizontal asymptotes)

Asymptotes

Vertical asymptote: $\boldsymbol{x = a}$ is a **vertical asymptote** of f if either

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \infty$$

(or $-\infty$) (or $-\infty$) (or $-\infty$)

Horizontal asymptote: $\boldsymbol{y = b}$ is a **horizontal asymptote** of f if

either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$

What does $\lim_{x \to \infty} f(x)$ mean?

$\lim_{x \to \infty} f(x) = L$ means

“the value of the function gets closer to L as we make x larger and larger (positive)”

$\lim_{x \to -\infty} f(x) = L$ means

“the value of the function gets closer to L as we make x larger and larger (negative)”

If the value of the function grows without bound as as we make x larger and larger, we say $\lim_{x \to \infty} f(x) = \infty$

There exists a function f such that $f(x) > 0, f'(x) < 0, f''(x) > 0$ for all x.

A) True

B) False

There exists a function f such that $f(x) > 0, f'(x) < 0, f''(x) > 0$ for all x.

A) True B) False

Then

$A)\ \lim_{x\to+\infty} f(x) = 0$

$B)\ \lim_{x\to+\infty} f(x) = L > 0$

$C)\ \lim_{x\to+\infty} f(x) = +\infty$

Suppose f is continuous for all x and

decreasing if $x < -1$ and increasing for $x > -1$,

concave up if $x < 0$ and concave down if $x > 0$.

Then $\lim_{x \to \infty} f(x) = \infty$.

A. True, I'm confident.
B. True, I'm not confident.
C. False, I'm confident.
D. False, I'm not confident.

How to evaluate limits at infinity

Building blocks: $\lim_{x \to \infty} c = c$ $\lim_{x \to -\infty} c = c$

$\lim_{x \to \infty} \frac{1}{x} = 0$ $\lim_{x \to -\infty} \frac{1}{x} = 0$

Limit laws

If $\lim_{x \to \infty} f(x) = F$ and $\lim_{x \to \infty} g(x) = G$ exist, then

$$\lim_{x \to \infty} f(x) \pm g(x) = F \pm G$$

$$\lim_{x \to \infty} f(x)g(x) = FG$$

$$\lim_{x \to \infty} f(x)/g(x) = F/G \quad \text{provided } G \neq 0$$

$$\lim_{x \to \infty} [f(x)]^r = F^r$$

Polynomials: Highest degree term dominates

e.g.

$$\lim_{x \to +\infty} x^2 - 4x + 100 = +\infty \qquad \lim_{x \to -\infty} x^2 - 4x + 100 = +\infty$$

$$\lim_{x \to +\infty} x^3 - 4x + 100 = +\infty \qquad \lim_{x \to -\infty} x^3 - 4x + 100 = -\infty$$

Rational functions: compare powers

e.g.

$$\lim_{x \to +\infty} \frac{3x^5 + x^2 - 5}{x^5 - 1}$$

$$\lim_{x \to +\infty} \frac{3x^4 + x^2 - 5}{x^5 - 1}$$

$$\lim_{x \to +\infty} \frac{x^5 - 1}{3x^4 + x^2 - 5}$$

Rational functions: compare powers

e.g.

$$\lim_{x\to+\infty}\frac{3x^5+x^2-5}{x^5-1}=\lim_{x\to+\infty}\frac{x^5(3+\frac{1}{x^3}-\frac{5}{x^5})}{x^5(1-\frac{1}{x^5})}=\lim_{x\to+\infty}\frac{(3+\frac{1}{x^3}-\frac{5}{x^5})}{(1-\frac{1}{x^5})}=3$$

$$\lim_{x\to+\infty}\frac{3x^4+x^2-5}{x^5-1}=\lim_{x\to+\infty}\frac{x^4(3+\frac{1}{x^2}-\frac{5}{x^4})}{x^5(1-\frac{1}{x^5})}=\lim_{x\to+\infty}\frac{(3+\frac{1}{x^2}-\frac{5}{x^4})}{x(1-\frac{1}{x^5})}=0$$

$$\lim_{x\to+\infty}\frac{x^5-1}{3x^4+x^2-5}=\lim_{x\to+\infty}\frac{x^5(1-\frac{1}{x^5})}{x^4(3+\frac{1}{x^2}--\frac{5}{x^4})}=\lim_{x\to+\infty}\frac{x(1-\frac{1}{x^5})}{(3+\frac{1}{x^2}-\frac{5}{x^4})}=+\infty$$

Careful with (even) roots....recall $\sqrt{x^2} = |x| = f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$

e.g. $\lim\limits_{t \to -\infty} \frac{\sqrt{3t^2+1}}{2t-1} = \lim\limits_{t \to -\infty} \frac{-t\sqrt{3+1/t^2}}{t(2-\frac{1}{t})} = \lim\limits_{t \to -\infty} \frac{-\sqrt{3+1/t^2}}{(2-\frac{1}{t})} = -\frac{\sqrt{3}}{2}$

Careful with indeterminate form $\infty - \infty$

e.g. $\lim\limits_{x \to +\infty} (x - \ln x) = \lim\limits_{x \to +\infty} x(1 - \frac{\ln x}{x}) = +\infty$

A variation of L'Hopital's rule

If $\lim_{x \to \infty} f(x) = \infty$ and $\lim_{x \to \infty} g(x) = \infty$ and f and g are both differentiable,

Then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$.

e.g. $\lim_{x \to +\infty} xe^{-x} = \lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$

$$\lim_{x \to +\infty} \frac{\ln x}{x} = \lim_{x \to +\infty} \frac{1/x}{1} = 0$$

Find all asymptotes (vertical and horizontal, if they exist) of the following functions;

$$f(x) = x^2 e^x$$

$$g(x) = xe^{-x^2}$$

$$h(x) = xe^{-1/x}$$

$$y(x) = \frac{\ln x}{x^2 - 1}$$

$$z(x) = \frac{\ln x}{x^2}$$