Science One Math

Nov 22, 2018

Shape of a curve

What kind of information about f can we can extract from the derivatives?

- the **first derivative** tells us about (rate of) **change in function values** (do they increase or decrease when *x* increases?)
- the second derivative tells us about (rate of) change in slope (does the slope increase or decrease?)
- derivatives allow us to locate local extreme values of a function
 - First-derivative test
 - Second-derivative test

Summary of tests

Increasing/Decreasing test

If f'(x) > 0 for all x in \mathcal{I} , f is increasing on \mathcal{I} . If f'(x) < 0 for all x in \mathcal{I} , f is decreasing on \mathcal{I} .

First derivative test for extrema

If x = c is a **critical number** of f and:

• f'(x) > 0 for x < c and f'(x) < 0 for $x > c \Rightarrow f(c)$ is a maximum • f'(x) < 0 for x < c and f'(x) > 0 for $x > c \Rightarrow f(c)$ is a minimum

Second derivative test for extrema

If x = c is a **critical number** of f and $f''(c) < 0 \Rightarrow f(c)$ is a **maximum**. If x = c is a **critical number** of f and $f''(c) > 0 \Rightarrow f(c)$ is a **minimum**. Let's test these ideas....

If f(p) is not a local extremum of f, then x = p is not a critical point of f.

- A. True, I'm confident.
- B. True, I'm not confident.
- C. False, I'm confident.
- D. False, I'm not confident.

Let f be a twice differentiable function everywhere, then f' is maximum at a point of inflection of f.

- A. True, I'm confident.
- B. True, I'm not confident.
- C. False, I'm confident.
- D. False, I'm not confident.

Consider two functions f and g, both differentiable for all x. Suppose f(x) is increasing for all x and g(x) is decreasing for all x. Then f(g(x)) is also increasing for all x.

- A. True, I'm confident.
- B. True, I'm not confident.
- C. False, I'm confident.
- D. False, I'm not confident.

Shape of curve

Useful information to sketch the graph of a function

- Domain
- Interval of increase/decrease
- Concavity
- Local extrema
- Inflection points
- Other useful points: intercepts, points at which $f(x) \rightarrow \infty$ or $f(x) \rightarrow -\infty$ (vertical asymptotes)
- End behaviour of f(x) as $x \to +\infty$ and $x \to -\infty$ (horizontal asymptotes)

Asymptotes

<u>Vertical asymptote</u>: x = a is a **vertical asymptote** of f if either $\lim_{x \to a} f(x) = \infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \infty$ (or $-\infty$) (or $-\infty$) (or $-\infty$)

<u>Horizontal asymptote</u>: y = b is a horizontal asymptote of f if either $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$

What does
$$\lim_{x\to\infty} f(x)$$
 mean?

 $\lim_{x \to \infty} f(x) = L \quad \text{means}$

"the value of the function gets closer to L as we make x larger and larger (positive)"

 $\lim_{x \to -\infty} f(x) = L \quad \text{means}$

"the value of the function gets closer to L as we make x larger and larger (negative)"

If the value of the function grows without bound as as we make x larger and larger, we say $\lim_{x\to\infty} f(x) = \infty$

There exists a function f such that f(x) > 0, f'(x) < 0, f''(x) > 0 for all x.

A) True B) False

There exists a function f such that f(x) > 0, f'(x) < 0, f''(x) > 0 for all x.

A) True B) False

Then

A) $\lim_{x \to +\infty} f(x) = 0$ B) $\lim_{x \to +\infty} f(x) = L > 0$ C) $\lim_{x \to +\infty} f(x) = +\infty$ Suppose f is continuous for all x and decreasing if x < -1 and increasing for x > -1, concave up if x < 0 and concave down if x > 0.

Then
$$\lim_{x\to\infty} f(x) = \infty$$
.

- A. True, I'm confident.
- B. True, I'm not confident.
- C. False, I'm confident.
- D. False, I'm not confident.

How to evaluate limits at infinity

Building blocks:
$$\lim_{x \to \infty} c = c$$
$$\lim_{x \to -\infty} c = c$$
$$\lim_{x \to -\infty} \frac{1}{x} = 0$$
$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

Limit laws

If $\lim_{x \to \infty} f(x) = F$ and $\lim_{x \to \infty} g(x) = G$ exist, then $\lim_{x \to \infty} f(x) \pm g(x) = F \pm G$ $\lim_{x \to \infty} f(x)g(x) = FG$ $\lim_{x \to \infty} f(x)/g(x) = F/G$ provided $G \neq 0$ $\lim_{x \to \infty} [f(x)]^r = F^r$

Polynomials: Highest degree term dominates

e.g.

$$\lim_{x \to +\infty} x^2 - 4x + 100 = +\infty \quad \lim_{x \to -\infty} x^2 - 4x + 100 = +\infty$$

 $\lim_{x \to +\infty} x^3 - 4x + 100 = +\infty \quad \lim_{x \to -\infty} x^3 - 4x + 100 = -\infty$

Rational functions: compare powers

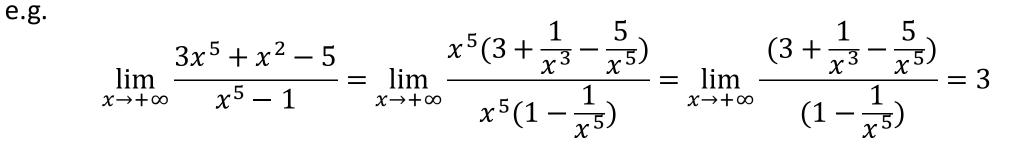
e.g.

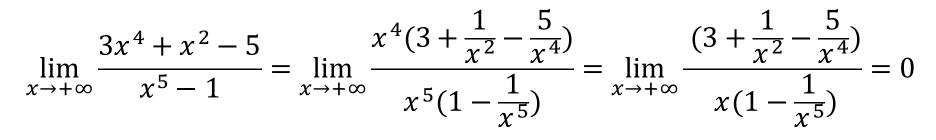
 $\lim_{x \to +\infty} \frac{3x^5 + x^2 - 5}{x^5 - 1}$

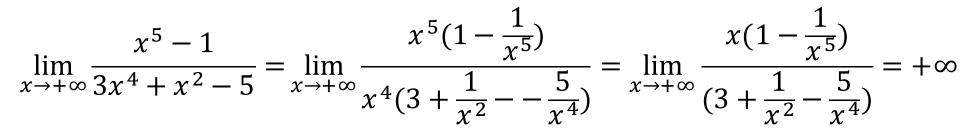
 $\lim_{x \to +\infty} \frac{3x^4 + x^2 - 5}{x^5 - 1}$

 $\lim_{x \to +\infty} \frac{x^5 - 1}{3x^4 + x^2 - 5}$

Rational functions: compare powers







Careful with (even) roots....recall
$$\sqrt{x^2} = |x| = f(x) = \begin{cases} -x, \ x < 0 \\ x, \ x \ge 0 \end{cases}$$

e.g.
$$\lim_{t \to -\infty} \frac{\sqrt{3t^2 + 1}}{2t - 1} = \lim_{t \to -\infty} \frac{-t\sqrt{3 + 1/t^2}}{t(2 - \frac{1}{t})} = \lim_{t \to -\infty} \frac{-\sqrt{3 + 1/t^2}}{(2 - \frac{1}{t})} = -\frac{\sqrt{3}}{2}$$

Careful with indeterminate form $\infty - \infty$

e.g.
$$\lim_{x \to +\infty} (x - \ln x) = \lim_{x \to +\infty} x(1 - \frac{\ln x}{x}) = +\infty$$

A variation of L'Hopital's rule

If
$$\lim_{x \to \infty} f(x) = \infty$$
 and $\lim_{x \to \infty} g(x) = \infty$ and f and g are both differentiable,
Then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$.

e.g.
$$\lim_{x \to +\infty} x e^{-x} = \lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

$$\lim_{x \to +\infty} \frac{\ln x}{x} = \lim_{x \to +\infty} \frac{1/x}{1} = 0$$

Find all asymptotes (vertical and horizontal, if they exist) of the following functions;

 $f(x) = x^2 e^x$

$$g(x) = xe^{-x^2}$$

$$h(x) = xe^{-1/x}$$

$$y(x) = \frac{\ln x}{x^2 - 1}$$

$$z(x) = \frac{\ln x}{x^2}$$