Science One Mathematics

September 18, 2018

Last time: How to evaluate limits

- Use basic limits and the limit laws, when they apply…
- If the limit laws don't apply, different strategies are possible.

When the limit laws fail: How to handle indeterminate form $\frac{0}{0}$

- Rewrite function in simplified form
  - Factor and cancel, e.g. $\lim_{u \to 1^+} \frac{(1-u^2)(1+u)}{1-2u+u^2}$
  - Multiply by conjugate, e.g. $\lim_{x \to 2} \frac{\sqrt{x+2}-2}{x-2}$
  - Expand parenthesis and simplify, e.g. $\lim_{h \to 0} \frac{(2+h)^2-4}{h}$
- Split the limit into right-hand and left-hand limits, e.g. $\lim_{y \to 3} \frac{y-3}{|y-3|}$
- (later in the term we'll apply l'Hopital's rule)

Another case when the limit laws are not useful…

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x^2}\right)$$

A. DNE because sine oscillates around 0.

B. DNE because $\frac{1}{x^2}$ is undefined at $x = 0$.

C. equals $(\lim_{x \to 0} x^2)(\lim_{x \to 0} \sin\left(\frac{1}{x^2}\right)) = 0$.

D. equals $\sin(1)$.

E. None of the above.

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A useful theorem

Squeeze Theorem

Consider three functions such that

$f(x) \leq g(x) \leq h(x)$ for all x near a point a

(except perhaps at $x = a$).

If $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = \mathrm{L}$, then $\lim_{x \to a} g(x) = L$

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right) = ?$$

We observe that $-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$ for all $x \neq 0$.

For all $x \neq 0, x^2 > 0$. It follows $-x^2 \leq x^2 \sin\left(\frac{1}{x^2}\right) \leq x^2$.

Since $\lim_{x\to 0} x^2 = 0 = \lim_{x\to 0} -x^2$, **by the squeeze theorem** it follows

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right) = 0.$$

Continuous Functions

Defn: A function is **continuous** at $x = a$

if $f(a)$ exists and $\lim_{x \to a} f(x) = f(a)$.

Implications:

- $f(a)$ is defined
- $\lim_{x \to a} f(x)$ exists (right and left limits are the same)
- the value of the limit equals the value of the function

4 types of discontinuity:

1. hole (can be eliminated)
2. jump
3. vertical asymptote
4. (infinite oscillations)

Which of the following functions is **NOT continuous** at $x = 0$?

$A)\ g(x) = x|x|$

$$B)\ f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$C)\ h(x) = \frac{x}{|x|}$

D) Both A and C

E) All three A, B, C.

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B) $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

C) $\boldsymbol{h(x) = \frac{x}{|x|}}$

D) Both A and C

E) All three A, B, C.

You decide to estimate e^2 by squaring longer decimal approximations of $e = 2.71828...$

A. This is a good idea because e is a rational number.
B. This is a good idea because $y = x^2$ is a continuous function
C. This is a good idea because $y = e^x$ is a continuous function.
D. This is a bad idea because e is irrational.

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Problem

The gravitational attraction of the earth on a mass m at a distance r from the centre of the earth is a continuous function $F(r)$ defined for $r \geq 0$ by

$$F(r) = \begin{cases} \dfrac{mgR^2}{r^2} & if\ \ r \geq R \\ mkr & if\ 0 \leq r < R \end{cases}$$

Find the constant k. (December Exam 2017)

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Find the constant *k*. (December Exam 2017)

$F(r)$ has possibly a discontinuity at $x = R$. To eliminate the discontinuity, we require $\lim_{r \to R^-} F(r) = \lim_{r \to R^+} F(r)$, that is $\lim_{r \to R^-} mkr = \lim_{r \to R^+} \frac{mgR^2}{r^2}$, $mkR = \frac{mgR^2}{R^2}$, thus $k = \frac{g}{R}$.

Two useful properties of continuous functions

- Intermediate Value property
- Extreme Value property (will discuss this later in the term)

Intermediate Value Theorem

If f is continuous on the interval $[a, b]$ and N is any value between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$,

then

there is a number c between a and b such that $f(c) = N$.

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there is a number c between a and b such that $f(c) = N$.

- In other words, f must cross the horizontal line $y = N$ <u>at least once</u> in the interval $[a, b]$.
- If $N = 0$, the intermediate value property implies that f has at least one *root* $x = c$ in $[a, b]$.

For the function $f(x) = x^2 - \frac{9}{x} + 1$, over which of the following intervals does the IVT guarantee a root?

A. [-3, -1]
B. [-1, 1]
C. [1, 3]
D. Both A and C
E. All three intervals

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Problem (2015 October midterm)

Show there are at least two solutions to $\frac{x}{2^x} = \frac{1}{3}$ for $0 \leq x \leq 4$.

A more challenging problem:

Show that along the equator there are two diametrically opposite sites that have exactly the same temperature at the same time.

Problem (2015 October midterm)

Show there are at least two solutions to $\frac{x}{2^x} = \frac{1}{3}$ for $0 \le x \le 4$.

Let $f(x) = \frac{x}{2^x} - \frac{1}{3}$. We observe $f(x)$ is continuous on $[0, 4]$, and $f(0) = -\frac{1}{3} < 0$ and $f(4) = \frac{1}{4} - \frac{1}{3} < 0$ and $f(1) = \frac{1}{2} - \frac{1}{3} > 0$, thus by IVT there must a number c_1 in $[0,1]$ such that $f(c_1) = 0$ and a number c_2 in $[1,4]$ such that $f(c_2) = 0$. It follows the original equation has at least two solutions $x = c_1$ and $x = c_2$ in $[0, 4]$.

Problem

Show that along the equator there are two diametrically opposite sites that have exactly the same temperature at the same time.

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Solution

Suppose $T(\theta)$ gives the temperature at a point on the equator with angle θ. Note $T(\theta)$ is a periodic and continuous on $[0, 2\pi]$, that is $T(0) = T(2\pi)$. We need to prove the equation $T(\theta) = T(\theta + \pi)$ has at least one solution.

Consider the function $f(\theta) = T(\theta) - T(\theta + \pi)$. We observe $f(\theta)$ is also periodic and continuous on $[0, 2\pi]$. In particular, on $[0, \pi]$ we have $f(0) = T(0) - T(\pi)$ and

$$f(\pi) = T(\pi) - T(2\pi) = T(\pi) - T(0) = -f(0)$$

Suppose $f(0) > 0$, then $f(\pi) < 0$. By IVT, there is a value θ_* in $[0, \pi]$ such that $f(\theta_*) = 0$. It follows $T(\theta_*) = T(\theta_* + \pi)$.