Science One Mathematics

September 20, 2018

Today

- Some practice with the Intermediate Value Theorem
- Mathematical definition of derivative and its interpretations

Intermediate Value Theorem

If f is continuous on the interval $[a, b]$ and N is any value between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, **then** there is a number c between a and b such that $f(c) = N$.

- In other words, f must cross the horizontal line $y = N$ <u>at least once</u> in the interval $[a, b]$.
- If $N = 0$, the intermediate value property implies that f has at least one *root* $x = c$ in $[a, b]$.

Problem: Show that $x^5 - x^4 - 4x - 1 = 0$ has at least three solutions in $\mathcal{R}$.

Solution

Let $f(x) = x^5 - x^4 - 4x - 1$, continuous everywhere because it's a polynomial. Consider the function values

x	-2	-1	0	1	2
$f(x)$	-41	1	-1	-5	7

By IVT,

given $f(-2) < 0 < f(-1)$, there exists one root c_1 in $(-2, -1)$,

given $f(0) < 0 < f(-1)$, there exists one root c_2 in $(-1, 0)$,

given $f(1) < 0 < f(2)$, there exists one root c_3 in $(1, 2)$.

Problem (2015 October midterm)

Show there are at least two solutions to $\frac{x}{2^x} = \frac{1}{3}$ for $0 \leq x \leq 4$.

A more challenging problem

Show that along the equator there are two diametrically opposite sites that have exactly the same temperature at the same time.

Problem (2015 October midterm)

Show there are at least two solutions to $\frac{x}{2^x} = \frac{1}{3}$ for $0 \leq x \leq 4$.

Let $f(x) = \frac{x}{2^x} - \frac{1}{3}$. We observe $f(x)$ is continuous on $[0, 4]$, and $f(0) = -\frac{1}{3} < 0$ and $f(4) = \frac{1}{4} - \frac{1}{3} < 0$ and $f(1) = \frac{1}{2} - \frac{1}{3} > 0$, thus by IVT there must a number c_1 in $(0,1)$ such that $f(c_1) = 0$ and a number c_2 in $[1,4]$ such that $f(c_2) = 0$. It follows the original equation has at least two solutions $x = c_1$ and $x = c_2$ in $(0, 4)$.

Problem

Show that along the equator there are two diametrically opposite sites that have exactly the same temperature at the same time.

Problem

Show that along the equator there are two diametrically opposite sites that have exactly the same temperature at the same time.

Solution

Suppose $T(\theta)$ gives the temperature at a point on the equator with angle θ. Note $T(\theta)$ is a periodic and continuous on $[0, 2\pi]$, that is $T(0) = T(2\pi)$. We need to prove the equation $T(\theta) = T(\theta + \pi)$ has at least one solution.

Consider the function $f(\theta) = T(\theta) - T(\theta + \pi)$. We observe $f(\theta)$ is also periodic and continuous on $[0, 2\pi]$. In particular, on $[0, \pi]$ we have

$$f(0) = T(0) - T(\pi)$$

$$f(\pi) = T(\pi) - T(2\pi) = T(\pi) - T(0) = -f(0)$$

Suppose $f(0) > 0$, then $f(\pi) < 0$. By IVT, there is a value θ_* in $(0, \pi)$ such that $f(\theta_*) = 0$. It follows $T(\theta_*) = T(\theta_* + \pi)$.

The derivative: Definition

The velocity problem

Given a position function $f(t)$, the **instantaneous velocity** at $t = a$ is

$$\boxed{\lim_{t \to a} \frac{f(t) - f(a)}{t - a} = \lim_{\delta t \to 0} \frac{\delta f}{\delta t}}$$

This is the limit of a difference quotient of the form 0/0.

Problem: Consider the position function (in m) $f(t) = 5t^2$, compute the velocity at $t = 2$ s.

Find the velocity function $v(t)$. [without using any differentiation rules]

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Problem: Consider the position function (in m) $f(t) = 5t^2$, compute the velocity at $t = 2$ s.

Solution: $\mathrm{v} = \lim_{t \to 2} \frac{5t^2 - 5 \cdot 2^2}{t - 2} = \lim_{t \to 2} \frac{5(t-2)(t+2)}{t - 2} = \lim_{t \to 2} 5(t + 2) = 20$ m/s

Find the velocity function $v(t)$. [without using any differentiation rules]

The velocity problem

Given a position function $f(t)$, the instantaneous velocity at $t = a$ is

$$\lim_{t \to a} \frac{f(t) - f(a)}{t - a} = \lim_{\delta t \to 0} \frac{\delta f}{\delta t}$$

Let $h = t - a$. When $t \to a$, $h \to 0$, the limit above becomes $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

The velocity $v(t)$ of any instant t is given by

$$\boxed{v(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h},}$$

this is a function of t.

Problem: Consider the position function (in m) $f(t) = 5t^2$, find the velocity function (in m/s).

The velocity problem

Given a position function $f(t)$, the instantaneous velocity at $t = a$ is

$$\lim_{t\to a}\frac{f(t)-f(a)}{t-a} = \lim_{\delta t\to 0}\frac{\delta f}{\delta t}$$

Let $h = t - a$. When $t \to a$, $h \to 0$, the limit above becomes $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

The velocity $v(t)$ of any instant t is given by

$v(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$ a function of t.

Problem: Consider the position function (in m) $f(t) = 5t^2$, find the velocity function (in m/s).

Solution:

$$v(t) = \lim_{h\to 0}\frac{5(t+h)^2-5\cdot t^2}{h} = \lim_{h\to 0}\frac{5t^2+10th+h^2-5\cdot t^2}{h} = \lim_{h\to 0}(10t+h) = 10t \quad \text{m/s}$$

The derivative: Definition

The **derivative** of a function **at a point** $t = a$ is defined as

$$\lim_{\delta t \to 0} \frac{\delta f}{\delta t} = \lim_{t \to a} \frac{f(t) - f(a)}{t - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a).$$

Mathematically, this means we can make the quotient $\frac{\delta f}{\delta t}$ be as close as we like to the number $f'(a)$ by taking values of h (or δt) sufficiently close to 0 (or values of t sufficiently close to a).

The **derivative function** is defined as $\lim_{h \to 0} \frac{f(t+h) - f(t)}{h} = f'(t)$.

Useful formulae to memorize for the quiz

Derivatives of Elementary Functions

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Basic Differentiation Rules

Power rule $\frac{d}{dx}[x^n] = nx^{n-1}$

Product rule $\frac{d}{dx}[f(x)\cdot g(x)] = \frac{df}{dx}\cdot g + f\cdot\frac{dg}{dx} = f'g + fg'$

Quotient rule $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'\cdot g - f\cdot g'}{[g]^2}$

Chain rule $\frac{d}{dx}[f(g(x)] = \frac{df}{dg}\cdot\frac{dg}{dx} = f(g(x))'\cdot g(x)'$