Science One Mathematics

September 25, 2018

The derivative: Definition

The **derivative** of a function $f(x)$ **at a point** $x = a$ is defined as

$$\lim_{\delta x \to 0} \frac{\delta f}{\delta x} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

The derivative at a point is denoted by $f'(a)$. This is a **number**.

If we allow the point to vary, we can build the **derivative function** defined as

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

The derivative: Interpretations

$f'(a)$ → **rate of change** of f at the point $x = a$.

$f'(a)$ → **slope of tangent** line to graph of $f(x)$ at $(a, f(a))$.

one number, two intepretations!

The equation of **tangent line** to graph of f at $(a, f(a))$ is

$$y = f(a) + f'(a)(x - a)$$

The derivative: Notation

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Different notation, same concept

$f'(x)$ $\quad \frac{df}{dx}$ $\quad \frac{d}{dx} f(x)$ $\quad Df(x)$ $\quad D_x f$ $\quad \dot{f}$

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Solution: Let (t, t^2) be the tangency point. The tangent line has slope $m = 2t$. The tangent line also goes through (a, b) so its slope must be $m = \frac{t^2 - b}{t - a}$. Thus, *t* must satisfy

$$2t = \frac{t^2 - b}{t - a}$$

$$t^2 - 2at + b = 0.$$

This equation has two solutions if $(2a)^2 - 4b > 0$, that is, if $a^2 - b > 0$, which is a true condition. Thus, from each point below the parabola there are two lines tangent to the parabola going through that point. There are no tangent lines going through any point above the parabola.

When is f differentiable?

A function is **differentiable** at $x = a$ if $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

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Examples of functions that fail to be differentiable at 0:

$\frac{1}{x}$ not differentiable at $x = 0$ because $f(0)$ DNE.

|x| not differentiable at $x = 0$ because left and right limits of the difference quotient are different.

$\sqrt{x}$ not differentiable at $x = 0$ because the limit of difference quotient diverges to infinity.

Question: Let $f(x) = x|x|$. Then

A. $f\,'(0) = 0$.

B. $f\,'(0)$ DNE because $|x|$ is not differentiable at $x = 0$.

C. $f\,'(0)$ DNE because f is defined piecewise.

D. $f\,'(0)$ DNE because the left and right derivative limits do not agree.

E. $f\,'(0)$ DNE because f is not continuous at $x = 0$.

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What does being differentiable at a point imply for f?

If $f\,'(a)$ exists, we know $f(a)$ exists and $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists. What about $\lim_{x \to a} f(x)$?

Think about this: If $f\,'(a)$ exists, then $\lim_{x \to a} f(x)$

A. must exist, but there is not enough information to determine it exactly.

B. equals $f(a)$.

C. equals $f\,'(a)$.

D. it may not exist.

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Does differentiability imply continuity?

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Proof: Start with $\lim_{x \to a} f(x)$. Add zero and multiply by 1.

$$\lim_{x \to a} f(x) = \lim_{x \to a} [f(x) + f(a) - f(a)] \frac{(x-a)}{(x-a)} =$$

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} (x - a) + f(a) \frac{(x-a)}{(x-a)} = f(a)$$

f is **continuous** at $x = a$. Yes, differentiability implies continuity!

The statement

“If $\boldsymbol{f}$ is differentiable at $\boldsymbol{x = a}$, $\boldsymbol{f}$ is continuous at $\boldsymbol{x = a}$”

means

A. if f is not continuous at a, f is not differentiable at a.
B. if f is not differentiable at a, f is not continuous at a.
C. Both A and B.

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Note that if f is continuous at $x = a$, we don’t know if it is also differentiable at $x = a$.

Continuity is a **necessary** condition for differentiability, but **not sufficient**. (Think $|x|$ at $x = 0$)

Problem (Midterm 2015)

$$f(x) = \begin{cases} e^{-x} & if\ x < 0 \\ ax + b & if\ 0 \le x \le 1 \\ x^2 - 1 & if\ x > 1 \end{cases}$$

i) Find a and b that make f continuous everywhere.

ii) With the values you found above, at which points is f differentiable?

iii) Find a function g which is defined and differentiable everywhere that is equal to f if $x < 0$ or $x > 1$.

‘Then you should say what you mean,’ the March Hare went on.

‘I do,’ Alice hastily replied; ‘at least–at least I mean what I say–that’s the same thing, you know.’

‘Not the same thing a bit!’ said the Hatter. ‘You might just as well say that “I see what I eat” is the same thing as “I eat what I see”!’

‘You might just as well say’, added the Dormouse, who seemed to be talking in his sleep, ‘that “I breathe when I sleep” is the same thing as “I sleep when I breathe”!’

‘It IS the same thing with you,’ said the Hatter, and here the conversation dropped.

“Alice’s Adventures in Wonderland”, Lewis Carroll