Science One Mathematics

September 2, 2018

When is f differentiable?

A function is **differentiable** at $x = a$ if $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

Examples of functions that fail to be differentiable at 0:

$\frac{1}{x}$ not differentiable at $x = 0$ because $f(0)$ DNE.

|x| not differentiable at $x = 0$ because left and right limits of the difference quotient are different.

$\sqrt{x}$ not differentiable at $x = 0$ because the limit of difference quotient diverges to infinity.

Question: Let $f(x) = x|x|$. Then

A. $f\,'(0) = 0$.

B. $f\,'(0)$ DNE because $|x|$ is not differentiable at $x = 0$.

C. $f\,'(0)$ DNE because f is defined piecewise.

D. $f\,'(0)$ DNE because the left and right derivative limits do not agree.

E. $f\,'(0)$ DNE because f is not continuous at $x = 0$.

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Solution. $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\sin\sqrt{|h|}-0}{h}$ since the denominator changes sign as h approaches 0 from the left or the right, we split the limit.

Recall $\sin\sqrt{|x|} = \begin{cases} \sin\sqrt{x} & \text{if} \quad x \geq 0 \\ \sin\sqrt{-x} & \text{if} \quad x < 0 \end{cases}$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$, then

$$\lim_{h\to 0^+} \frac{\sin\sqrt{h}}{h} = \lim_{h\to 0^+} \frac{\sin\sqrt{h}}{\sqrt{h}} \frac{1}{\sqrt{h}} = +\infty$$

$$\lim_{h\to 0^-} \frac{\sin\sqrt{-h}}{-|h|} = \lim_{h\to 0^-} \frac{\sin\sqrt{-h}}{-\sqrt{|h|^2}} = \lim_{h\to 0^-} \frac{\sin\sqrt{-h}}{\sqrt{-h}} \frac{1}{-\sqrt{-h}} = -\infty$$

What does being differentiable at a point imply for f?

If $f\,'(a)$ exists, we know $f(a)$ exists and $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists.

What about $\lim_{x \to a} f(x)$?

Think about this: If $f\,'(a)$ exists, then $\lim_{x \to a} f(x)$

A. must exist, but there is not enough information to determine it exactly.
B. equals $f(a)$.
C. equals $f\,'(a)$.
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Proof: Start with $\lim_{x \to a} f(x)$. Add zero and multiply by 1.

$$\lim_{x \to a} f(x) = \lim_{x \to a} [f(x) + f(a) - f(a)] \frac{(x-a)}{(x-a)} =$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} (x - a) + f(a) \frac{(x-a)}{(x-a)} = f(a)$$

f is **continuous** at $x = a$. Yes, differentiability implies continuity!

The statement

“If $\boldsymbol{f}$ is differentiable at $\boldsymbol{x = a}$, $\boldsymbol{f}$ is continuous at $\boldsymbol{x = a}$”

means

A. if f is not continuous at a, f is not differentiable at a.
B. if f is not differentiable at a, f is not continuous at a.
C. Both A and B.

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Note that if f is continuous at $x = a$, we don’t know if it is also differentiable at $x = a$.

Continuity is a **necessary** condition for differentiability, but **not sufficient**. (Think $|x|$ at $x = 0$.)

Problem (Midterm 2015)

$$f(x) = \begin{cases} e^{-x} & if\ x < 0 \\ ax + b & if\ 0 \leq x \leq 1 \\ x^2 - 1 & if\ x > 1 \end{cases}$$

i) Find a and b that make f continuous everywhere.

ii) With the values you found above, at which points is f differentiable?

iii) Find a function g which is defined and differentiable everywhere that is equal to f if $x < 0$ or $x > 1$.