Science One Math

October 16, 2018

Recap: How to compute a derivative

- Limit definition of derivative
  - for finding the derivative of elementary functions from first principles
  - determining whether a piecewise function is differentiable at the boundary point
  - for evaluating a limit of the form $\lim_{x \to a}[f(x) - f(a)]/(x - a)$
  - for proving statements involving some unknown function f and the above limit
- Differentiation rules (constant multiple, sum, difference, power, product, quotient, chain rule)
- Implicit differentiation (basically chain rule)
  - For finding the derivative of inverse functions
  - For finding the slope of the tangent line to an algebraic curve
  - For dealing with logarithmic differentiation, which is essential for finding the derivative of a function of the form $[f(x)]^{g(x)}$
  - For differentiating an equation between quantities that depend on the same variable

Derivative of a function $[f(x)]^{g(x)}$ (from Stephen's slides last Thursday)

$y = x^x \Longrightarrow \ln y = \ln x^x \Longrightarrow \ln y = x \ln x$ *now differentiate both sides w.r.t. x*

$$\frac{1}{y}\frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x} \quad \Longrightarrow \quad \frac{dy}{dx} = (\ln x + 1)y = (\ln x + 1)\,x^x.$$

Another example: Given $[\ln(y)]^y = x^2$, find $\frac{dy}{dx}$.

$y \ln(\ln y) = 2 \ln x$ *treat y as a function of x, applied the chain rule to y*

$$\frac{dy}{dx}\ln(\ln y) + y\frac{1}{\ln y}\frac{1}{y}\frac{dy}{dx} = \frac{2}{x}$$

factor out $\frac{dy}{dx}$ *and solve*

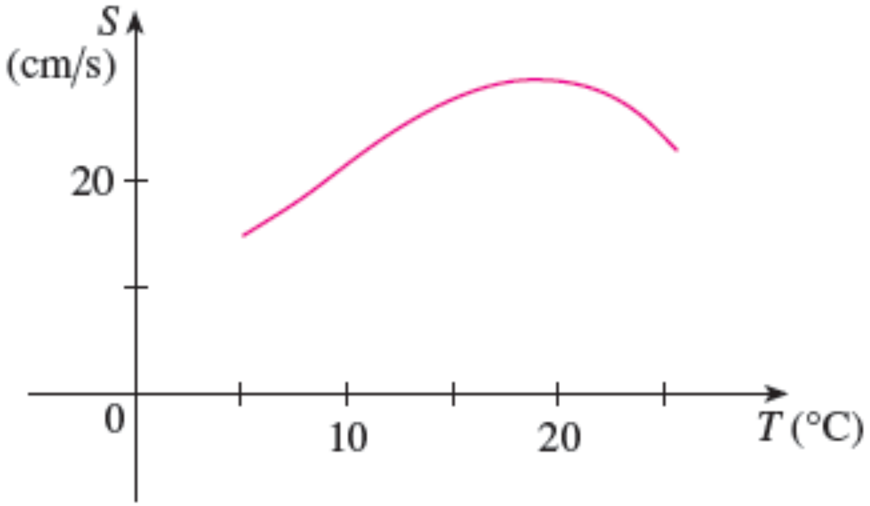
$$\frac{dy}{dx} = \frac{2}{x}\,\frac{1}{\ln(\ln y) + \frac{1}{\ln y}}$$

Interpretations of the derivative

The derivative of f at a point, $f'(x)$, represents:

- Slope of tangent line to the graph of f at that point.
- **Rate of change** of f at that point (with respect to a change in x)

Problem: This graph shows the influence of the temperature T on the maximum sustainable swimming speed S of Coho salmon. What does $S'(T)$ represent?



A. The rate at which the maximum sustainable speed of Coho salmon changes for a change in temperature

B. The maximum sustainable acceleration of Coho salmon at temperature T

C. The rate of change of temperature for a change in the maximum sustainable speed of Coho salmon

D. The change in maximum sustainable speed of Coho salmon for a given temperature.

Rates in science: examples

Problem: The value of velocity of blood v in an artery changes depending on the distance r from the artery walls (blood flows faster in the centre of the artery).

What quantity represents the rate at which the velocity of blood changes for a change in the distance from the artery wall at time t?

A) $\frac{dv}{dt}$

B) $\frac{dv}{dr}$

C) $\frac{dr}{dt}$

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In general, the derivative $\frac{dy}{dx}$ enables us to quantify how a quantity y is changing as x changes at a given x-value.

Rates in science: Related Rates

Question: If two quantities that are related (such as the side length and volume of a cube) are both changing as implicit functions of time, how are their rates of change related?

Given a relationship between A and B,

is there a relationship between $\frac{dA}{dt}$ *and* $\frac{dB}{dt}$ *?*

i.e. are the rates related?

Example: Suppose a cubic crystal is growing so that its side length changes at a rate of *r* (in mm/s) when it is L mm. How fast is the crystal growing in volume at that instant?

Rates in science: examples

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Solution: Our model for the crystal is a cube. At any time, its volume is $V = x^3$. Both V and x increase with time. By differentiating implicitly, we find the rate of change of volume is

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

When $x = L$, $\frac{dx}{dt} = r$, hence $\frac{dV}{dt} = 3\, L^2\, r$ mm^3/s.

Question: Suppose the side length grows at a constant rate, does the volume of the cube grow at a constant rate too?

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Solution: At any instant, the volume of the snowball is $V = \frac{4\pi}{3} r^3$.

By differentiating both sides w.r.t. time, we have

$$\frac{dV}{dt} = -4\pi r^2 \frac{dr}{dt} \implies \frac{dr}{dt} = -\frac{1}{4\pi r^2} \frac{dV}{dt} = -\frac{k}{4\pi r^2} \text{ cm/min.}$$

Geometrically, $D = 2r \implies \frac{dD}{dt} = 2\frac{dr}{dt} \implies$ when $D = a, r = \frac{a}{2}$ and

$$\frac{dD}{dt} = -\frac{2k}{\pi a^2} \text{ cm/min.}$$

Does the diameter of the snowball melt faster when it's bigger or smaller?

Problem: Gravel is being poured into a conical pike from a conveyor belt. As more gravel is added, the volume V of the pile changes over time, and so do the height h and radius r of the pile.

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Suppose gravel is added to the pile at a constant rate. Do you expect the height of the pile to grow at a constant rate?

A. Yes
B. No
C. Not sure

[if your answer is B, make a prediction of how the pile grows over time].

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A. Yes

B. No Let's see why.

C. Not sure

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Which of the following mathematical quantities represents the **rate at which gravel is added to the pile**?

Let V, h, and r be the volume height and radius of the pile at time t.

A. $\frac{dV}{dh}$ B) $\frac{dV}{dt}$ C) $\frac{dV}{dr}$ D) $\frac{dh}{dt}$ E) $\frac{dr}{dt}$

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Model the pile of gravel as a cone whose volume at a time t is $V = \frac{\pi}{3} r^2 h$, where h and r are the height and radius of the pile at time t.

Which of the following equations represents the relationship between the rate of change of volume and other relevant variables in the problem?

A. $\frac{dV}{dt} = \frac{\pi}{3}\left(2r \frac{dr}{dt} \cdot \frac{dh}{dt} \right)$

B. $\frac{dV}{dt} = \frac{\pi}{3}\left(2r \frac{dh}{dt} \right)$

C. $\frac{dV}{dt} = \frac{\pi}{3}\left(2r \frac{dr}{dt} h + r^2 \frac{dh}{dt} \right)$

D. $\frac{dV}{dt} = \frac{\pi}{3}\left(r^2 + 2r \frac{dr}{dh} h \right)$

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Suppose gravel is added to the pile at a constant rate of k m^3/min . How fast is the height of the pile growing when the pile is h m high?

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Solution: The rate of change of volume

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right)$$

We are given $\frac{dV}{dt} = k$ m^3/min, we seek $\frac{dh}{dt}$.

What is $\frac{dr}{dt}$?

are *r* and *h* related? depends on shape of pile!

We need more information.

Suppose gravel is added to the pile at a constant rate of k m³/min . How fast is the height of the pile growing when the pile is h m high?

Assume the proportions of the pile remain constant as the pile grows and the pile if $\boldsymbol{H}$ m high and $\boldsymbol{D}$ m wide (at the base) when the conveyor belt stops.

How fast is the height of the pile growing when the pile is h m high (with $h < H$)? When is $h/2$ m high?

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Follow up question: How fast should gravel be dumped from the conveyor belt to guarantee the height of the pile grows at a constant rate?

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Solution: Assume $\frac{h}{2r} = \frac{H}{D} \implies r = \frac{D}{2H} h \implies \frac{dr}{dt} = \frac{D}{2H}\frac{dh}{dt}$

hence

$$\frac{dV}{dt} = \frac{\pi}{4}\frac{D^2}{H^2} h^2 \frac{dh}{dt}$$

solve for $\frac{dh}{dt}$, given $\frac{dV}{dt} = k$.

$$\frac{dh}{dt} = \frac{4H^2}{\pi D^2}\frac{k}{h^2}$$

the height is not growing at a constant rate!

Follow-up: Express h as an explicit function of t .

Problem: Suppose we are filling a cylindrical tank and a conical tank (pointing downwards) with water at the same rate *k* m^3/min. The tanks have the same height and volume. Soon after we start pouring water into the tanks,

A. the water is rising at the same rate in both tanks
B. the water is rising faster in the conical tank
C. the water is rising faster in the cylindrical tank

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When will the water rise at the same rate in both tanks?

Problem: A lighthouse is L m from the straight shore on a rock. The light rotates at α rev/min. How fast is the light moving along the shore at a point X m from a point on the shore directly opposite the lighthouse?

Problem: A spider moves horizontally across the ground at a constant rate, *b*, pulling a thin silk thread with it. One end of the thread is tethered to a vertical wall at height *h* above ground and does not move. The other end moves with the spider. Find an expression for the rate of elongation of the thread in terms of *b* and *x*, the position of the spider at time *t*.