Science One Mathematics

*This exam has **10 questions** on **11 pages**, for a total of 76 points.*

Duration: 150 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; for changes of variables state how the variables are related. For integration by parts, state what the parts are. Answers without justifications will not be accepted.
- Continue on blank pages if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First name: ____________________ Last name: ____________________

Student #: ____________________ Bamfield #: ____________________

Signature: ____________________

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	6	12	8	8	6	6	10	8	6	6	76
Score:											

6 marks

1. This question has three independent parts.

(a) Suppose $f(x) \geq 0$, and

$$\int_{-2}^{2} f(x)dx = 4, \quad \int_{2}^{5} f(x)dx = 3, \quad \int_{-2}^{5} g(x)dx = 2.$$

Which, if any, of the following statements are true? Circle all that apply.

(A) $\int_5^2 f(x)dx = -3$

(B) $\int_{-2}^5 [f(x) + g(x)]dx = 9$

(C) $\int_{-2}^2 2f(x)dx = 8$

(D) $\int_{-2}^2 \sqrt{f(x)}dx = 2$

(E) The area under the graph of g on the interval $-2 \leq x \leq 5$ is 2 units.

(b) Suppose f is a function that is increasing, positive, and differentiable for all x. Let

$$g(x) = \int_0^x f(t)dt.$$

Which, if any, of the following statements are true? Circle all that apply.

(A) g is differentiable for all x.

(B) $g(x) \geq 0$ for all x.

(C) g is increasing and concave up.

(D) g has only one zero at $x = 0$.

(E) If $h(z) = \int_0^z f(t)dt$, then h and g are the same functions.

(c) Which, if any, of the following statements are always true? Circle all that apply.

(A) If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is a convergent series.

(B) If $\sum a_n$ converges, then $\sum(1/a_n)$ diverges.

(C) If $a_n > 0$ and $\sum a_n$ converges, then $\sum(-1)^n a_n$ converges.

(D) It's possible to have a power series whose interval of convergence is $[0, +\infty)$.

(E) The series $\displaystyle\sum_{n=0}^{300} 3^n + \sum_{n=301}^{\infty} (1/3)^n$ converges.

$\boxed{\text{12 marks}}$ 2. Compute the following integrals.

(a) $\displaystyle\int \sec^2(x)\,[1 + \sec(x)\tan(x)]\,dx$

(b) $\displaystyle\int \frac{t+3}{t^2-1}\,dt$

(c) $\int_1^2 \ln^2(y)\,dy$

$\boxed{\text{8 marks}}$ 3. Determine whether each series converges. Justify your answer, by stating which test you are using. You may use known facts about the convergence of geometric series and p-series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3n^2 + \sqrt{n}}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{(k!)^2}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{4^k}{3^k + 5^k}$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^{\frac{3}{2}}}$

8 marks

4. (a) By using operations on power series (e.g. substitution, integration, differentiation, etc.), find the power series representation (centred at 0) of the function

$$f(x) = \frac{x}{(1+x^2)^2}$$

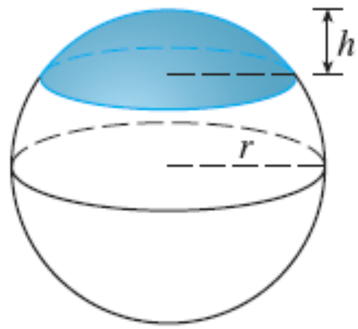
(you may just write the first 5 non-zero terms).

(b) Determine the interval of convergence of the series found in part (a).

(c) Find an infinite series expression (you may just write the first 5 non-zero terms) for

$$\int_0^1 \frac{x^2}{(1+x^4)^2} dx.$$

6 marks 5. Find the volume of the cap formed when a horizontal slice is made at a distance h from the top of a ball of radius r ($h < r$).



6 marks

6. (a) To design the interior surface of a huge stainless-steel tank, you revolve the curve $y = x^2$ for $0 \le x \le 4$ about the y-axis. The container (with dimensions in metres) is filled with seawater (of density ρ). How much work will it take to empty the tank by pumping the water to the tank's top? Let g be the acceleration due to gravity.

(b) Suppose the pump breaks down after w J of work has been done. Write down an equation whose solution gives the depth of the water remaining in the tank after the pump broke down.

10 marks

7. The time (in years) it takes for a certain radioactive atom to decay is a random variable with probability density function

$$f(t) = \begin{cases} 0 & t < 0 \\ ke^{-kt} & t \geq 0 \end{cases}$$

where k is a constant.

(a) What is the probability that the atom does *not* decay within t years?

(b) If the median decay time (half-life) is 10 years, find k. Recall that the median of a distribution is the value separating the higher half of the distribution from the bottom half.

(c) Find the mean decay time.

8 marks

8. The Von Bertalanffy growth model is used to predict the length $L(t)$ of an individual fish over a period of time t. If L_∞ (in centimetres) is the largest length for a species, then the hypothesis in this model is that the rate of growth in length is proportional to the difference between the current length L and the asymptotic length L_∞.

(a) Write a differential equation that expresses this idea.

(b) Solve the differential equation found above to find an explicit expression for $L(t)$ that satisfies $L(0) = a$ for $a > 0$.

6 marks 9. Determine (with justification) whether each of the improper integrals converges or diverges (careful: there is no elementary antiderivative for these integrals)

(a) $\displaystyle\int_0^1 \frac{e^x}{x} dx$

(b) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x+x^3}}\, dx$

$\boxed{\text{6 marks}}$ 10. (a) Find the sum of the series

$$1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 \frac{1}{2!} + \left(\frac{1}{2}\right)^3 \frac{1}{3!} + \left(\frac{1}{2}\right)^4 \frac{1}{4!} + \cdots + \left(\frac{1}{2}\right)^n \frac{1}{n!} + \ldots$$

(b) Find the sum of the series

$$1 + \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 3 + \cdots + \frac{n}{2^{n-1}} + \ldots$$