Science One Mathematics

*This exam has **10 questions** on **11 pages**, for a total of 76 points.*

Duration: 150 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; for changes of variables state how the variables are related. For integration by parts, state what the parts are. Answers without justifications will not be accepted.
- Continue on blank pages if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First name: ______________________ Last name: ______________________

Student #: ______________________ Bamfield #: ______________

Signature: ____

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	8	12	8	6	6	8	8	8	8	4	76
Score:											

8 marks

1. Determine whether each of the following statements is true or false. **Provide justification** (either an explanation or a counterexample).

   (a) **True/False**. $\frac{d}{dy}\int_0^y xf(u)dx = yf(y)$, where f is continuous for all reals.

   (b) **True/False**. $\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -2.$

   (c) **True/False**. $\sum_{n=0}^{\infty}(-1)^n(x-1)^n = \frac{1}{x}$ for $0 < x < 2$.

   (d) **True/False**. If $\sum_{n=0}^{\infty} a_n$ diverges then $\lim_{n\to\infty} a_n \neq 0$.

12 marks

2. Compute the following integrals.

(a) $\int_0^{\pi/2} \sqrt{1+\sin^2(x)}\,\sin(x)\cos(x)\;dx$

(b) $\int \frac{1}{t^2+2t+3} dt$

(c) $\int \frac{\ln(\ln(y))}{y}\;dy$

8 marks 3. Determine whether each series converges. Justify your answer, by stating which test you are using. You may use known facts about the convergence of geometric series and p-series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3+e^{-n}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3n^2+n}$

(c) $1+\dfrac{1}{3}+\dfrac{1}{5}+\dfrac{1}{7}+\dfrac{1}{9}+\cdots$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{k^{100}100^k}{k!}$

6 marks

4. (a) Find the area of the region to the right of the y-axis that is bounded by the graphs $y = x^2$ and $y = 6 - x$.

(b) Find the x-coordinate of the centroid of the region described in part (a).

6 marks 5. The profile of a tank is obtained by rotating the region described in the previous question (the region to the right of the y-axis that is bounded by the graphs $y = x^2$ and $y = 6 - x$) about the y-axis. What is the work done against gravity to fill the tank to the top with a fluid of density ρ (in $\mathrm{kg/m^3}$). Assume the fluid is taken from a reservoir at ground level. Let g be the acceleration due to gravity.

8 marks

6. Find the volume of the "elliptical doughnut" swept out when the area inside the ellipse $4(x-1)^2 + y^2 = 1$ is rotated about the y-axis.

8 marks 7. Suppose the distance $R \geq 0$ of a quantum particle from a certain point is a random variable described by the probability density function

$$f(r) = \frac{2}{\sqrt{\pi}} e^{-r^2}.$$

(a) Write (but do not evaluate) an integral giving the probability that the particle is a distance no more than 1 from the point.

(b) Find the mean distance of the particle from the point.

(c) Find an infinite series expression for the probability in part (a).

8 marks 8. Suppose the population $P(t)$ of some species, as a function of time t, is governed by an ODE initial value problem

$$\frac{dP}{dt} = -kP(P-A)(P-B), \quad P(0) = P_0,$$

for some constants $k > 0$, $0 < A < B$, and $P_0 \geq 0$.

(a) Without solving the equation, determine $\lim_{t\to\infty} P(t)$. Your answer will depend on the value of P_0.

(b) Give a brief biological interpretation of each of the constants A and B.

(c) If $k = 1$, $A = 1$, $B = 2$, $P_0 = \frac{3}{2}$, solve the ODE to obtain an implicit relation between P and t.

8 marks 9. Determine (with justification) whether each of the improper integrals converges or diverges. If it converges, compute its value.

(a) $\displaystyle\int_0^{\infty} \frac{1}{e^x + e^{-x}} dx.$

(b) $\displaystyle\int_{-\infty}^{+\infty} \frac{x+1}{x^2+1} dx.$

4 marks 10. Evaluate

$$\lim_{n\to\infty} \frac{\sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{3}+\cdots+\sqrt[3]{n}}{n^{4/3}}$$

by first finding a function f such that the limit is equal to

$$\int_0^1 f(x)dx.$$