*This midterm has **9 questions** on **11 pages**, for a total of 100 points.*

Duration: 150 minutes

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; for changes of variables state how the variables are related. For integration by parts, state what the parts are. Answers without justifications will not be accepted.
- Continue on blank pages if you run out of space.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First name: ______________________ Last name: ______________________

Student #: ______________________ Bamfield #: ______________

Signature: ____

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	15	15	9	6	9	10	12	12	12	100
Score:										

15 marks 1. This question has three independent parts.

(a) If f is continuous and $\int_0^9 f(x)dx = 4$, find $\int_0^3 xf(x^2)dx$.

(b) Find $f'(2)$ if $f(x) = e^{g(x)}$ and $g(x) = \displaystyle\int_2^x \frac{t}{1+t^4}dt$.

(c) A function $y = f(x)$ is always positive and satisfies the equation $y' = xy$ for all x. Find f such that $f(0) = e$.

15 marks 2. Compute the following integrals.

(a) $\int_0^1 \frac{x}{\sqrt{4-x^2}} dx$

(b) $\int_0^\infty \frac{dx}{(x+1)(x+3)}$

(c) $\int (x+1)e^{-2x} dx$

9 marks

3. Determine whether each series converges. Justify your answer.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

6 marks

4. Find the following limits:

   (a) $\lim_{n\to\infty} \frac{3^n}{n!}$

   (b) $\lim_{n\to\infty} (5n)^{1/n}$

9 marks 5. (a) Find the Taylor series centred at $x = 0$ for the function $J(x) = \displaystyle\int_0^x \frac{e^t - 1}{t} dt$.

(b) Determine the interval of convergence for the series obtained in part (a).

10 marks 6. (a) Compute the area of the shape bounded in the quadrant $x, y > 0$ by the curve $\sqrt{x}+\sqrt{y} = 1$.

(b) Compute the volume of a solid of revolution obtained by revolving the shape described above around the x-axis.

12 marks 7. (a) Alice and Bob wish to build a sand castle, in the shape of a cone, with radius 1m and height 3m. What is the total amount of work needed to lift the sand for the project? Gravity is g and the density of sand is ρ.

(b) If Alice constructs the foundation up to a height of 1 metre, and Bob finishes the construction, who does more physical work? Who uses a larger amount of sand? Justify your answer.

12 marks

8. The famous golfer *Cat Forests* hits a ball and it lands at a random distance X from the target, where X has probability density $2kxe^{-kx^2}$ for some constant k.

(a) Find a formula for the probability of the ball landing at distance at most L from the target.

(b) Cat hits within at most 10 metres 3/4 of the time. Find k.

(c) If a shot misses by more than 20 metres, it lands in the lake. Find the probability that Cat's ball lands in the water.

12 marks 9. A bag-pipe player blows air into the bag at a constant rate of 10 litres/minute. The bag has a constant volume of 20 litres. The piper had a bottle of Scotch (18 year old Clynelish), and so his breath contains 2% alcohol.

(a) Give a differential equation for the concentration of alcohol in the bag at time t.

(b) Find a solution to this equation.

(c) At what time does the concentration reach 1%?

(d) At what time does the concentration reach 2%?

Trigonometric identities For any θ:

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \qquad \cos^2\theta + \sin^2\theta = 1$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \qquad 1+\tan^2\theta = \sec^2\theta$$

Probability For a random variable taking values between a and b (possibly $a = -\infty$ and $b = \infty$):
Mean is $\mu = \int_a^b x f(x)\, dx$.
Standard deviation is σ where $\sigma^2 = \int_a^b (x-\mu)^2 f(x) dx$.
The median is M such that $\int_a^M f(x) dx = \frac{1}{2}$.

Generalized Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = b'(x)f(b(x)) - a'(x)f(a(x)).$$

Centre of mass $\overline{x} = \frac{M_x}{M}$, where $M = \int f(x)dx$ and $M_x = \int x f(x) dx$.

Taylor polynomials and series

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$f(x) = T_n(x) + R_n(x) \qquad \text{where} \qquad R_n(x) = \frac{f^{(n+1)}(\theta)}{(n+1)!}(x-a)^{n+1}$$

The following are series around $x = 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \qquad \frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \qquad \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Trigonometric substitutions The following are sometimes useful:
If $u = \tan(x/2)$ then $dx = \frac{2du}{1+u^2}$ and $\sin x = \frac{u^2}{1+u^2}$ and $\cos x = \frac{1-u^2}{1+u^2}$.
If $u = a\sin\theta$ then $du = a\cos\theta d\theta$ and $\sqrt{a^2-u^2} = \cos\theta$.