8 1. Determine whether each of the following statements is true or false. **Provide justification**.

(a) True/False. If f is continuous on $[a, b]$ then $\frac{d}{dx}\int_a^b f(t)dt = f(x)$.

> **Solution:** False. $\int_a^b f(t)dt$ is a number so its derivative is zero and does not depends on x.

(b) True/False. Suppose $f(x)$ is a continuous function for all x and such that $f(0) = 0$. Let $g(x) = \int_{-1}^x f(t)dt$. Then the graph of g is tangent to the x-axis at $x = 0$.

> **Solution:** False. For the graph of g to be tangent to the x-axis at $x = 0$ we need $g'(0) = 0$ and $g(0)$. The first condition is satisfied because $g'(x) = f(x)$ by the fundamental theorem of calculus, and we were told $f(0) = 0$ so $g'(0) = 0$. However, the second condition is not satisfied because $g(0) = \int_{-1}^0 f(t)dt$ may or may not be 0.

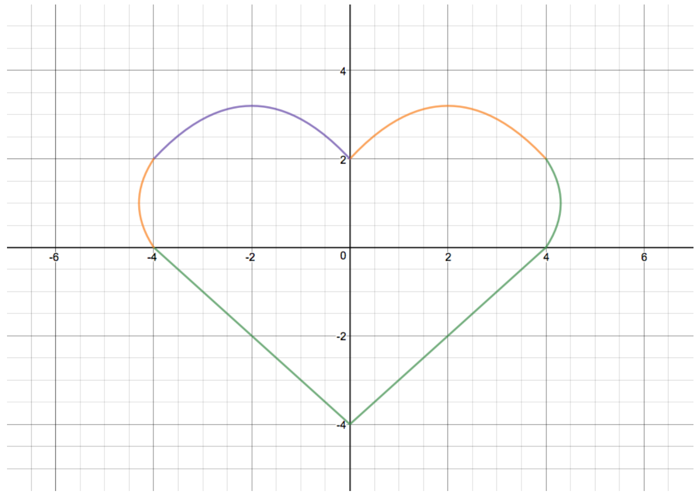
(c) True/False. If f has a discontinuity at $x = 0$, then $\int_{-1}^1 f(x)dx$ does not exist.

> **Solution:** False. If f has a removeable discontinuity, then f is integrable. If f has an infinite discontinuity, the given integral is improper but it could converge.

(d) True/False If f is continuous and $\int_0^9 f(x)dx = 4$, then $\int_0^3 xf(x^2)dx = 2$.

> **Solution:** True. Let $u = x^2$, then $du = 2xdx$ and the given integral becomes
> $$\int_0^3 xf(x^2)dx = 2 = \frac{1}{2}\int_0^9 f(u)du = \frac{1}{2} \cdot 4 = 2.$$

6 2. Happy Valentine's Day.



Compute the area of the region enclosed by the heart-shaped curve shown above. The curve is described by the following equations,

$$\begin{array}{ll} y = |x| - 4 & \text{if } -4 \leq x \leq 4 \\ y = 0.3x(4-x) + 2 & \text{if } 0 \leq x \leq 4 \\ y = -0.3x(4+x) + 2 & \text{if } -4 \leq x \leq 0 \\ x = -0.3y^2 + 0.6y + 4 & \text{if } 0 \leq y \leq 2 \\ x = 0.3y^2 - 0.6y - 4 & \text{if } 0 \leq y \leq 2. \end{array}$$

Solution: The heart-shaped region is symmetric with respect to the y-axis so we can compute the area of the right-end side of the heart and then multiply the result by 2. The area of the right-end side of the heart can be computed by splitting the region into two (or more) regions. The first part is the region below $y = 0.3x(4-x) + 2$ and above $y = |x| - 4$ for $0 \leq x \leq 4$, which has area

$$\int_0^4 0.3x(4-x) + 2 - (x-4)dx = \int_0^4 -0.3x^2 + 0.2x + 4dx =$$

$$= (-0.1x^3 + 0.1x^2 + 4x)_0^4 = 19.2.$$

The second part is the region bounded by $x = -0.3y^2 + 0.6y + 4$ and $x = 4$ for $0 \leq y \leq 2$, which has area

$$\int_0^2 (-0.3y^2 + 0.6y + 4 - 4)dy = (-0.1y^3 + 0.3y^2)_0^4 = 0.4.$$

So the total area is $2(19.2 + 0.4) = 39.2$.

20 3. Evaluate any **four** integrals of your choice from the list below. Remember to include integration constants whenever appropriate. Continue your work on the next page if you need more space.

(a) $\displaystyle\int \frac{1}{(x-1)(x^2+1)}\,dx$ (b) $\displaystyle\int \frac{\sin^3(x)}{\cos^2(x)}\,dx$ (c) $\displaystyle\int_0^3 (x+1)\sqrt{9-x^2}\,dx$

(d) $\displaystyle\int \frac{2x-1}{x^2-2x+5}\,dx$ (e) $\displaystyle\int_0^4 \log(1+x^2)\,dx$

Solution: (a)

$$\int \frac{1}{(x-1)(x^2+1)}\,dx = \frac{1}{2}\int \frac{1}{x-1}dx - \frac{1}{2}\int \frac{x+1}{x^2+1}dx =$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\int \frac{2x}{x^2+1}dx - \frac{1}{4}\int \frac{2}{x^2+1}dx =$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\arctan x + C.$$

(b)

$$\int \frac{\sin^3(x)}{\cos^2(x)}\,dx = \int \frac{\sin^2(x)\sin(x)}{\cos^2(x)}dx = \int \frac{(1-\cos^2(x))\sin(x)}{\cos^2(x)}dx =$$

$$= \int \frac{u^2-1}{u^2}du = \int 1 - \frac{1}{u^2}du = u + \frac{1}{u} + C = \cos(x) + \frac{1}{\cos(x)} + C.$$

(c)

$$\int_0^3 (x+1)\sqrt{9-x^2}\,dx = 9\int_0^{\pi/2} (3\sin\theta + 1)\cos^2\theta d\theta =$$

$$= 9\int_0^{\pi/2} 3\sin\theta\cos^2\theta d\theta + 9\int_0^{\pi/2} \cos^2\theta d\theta =$$

$$= -9\cos^3\theta|_0^{\pi/2} + 9\int_0^{\pi/2} \frac{1}{2}(\cos 2\theta + 1)d\theta = -9\cdot 0 + 9\cdot 1 + (\frac{9}{4}\sin 2\theta + \frac{9}{2}\theta)|_0^{\pi/2} = 9 + \frac{9\pi}{4}.$$

(d)

$$\int \frac{2x-1}{x^2-2x+5}dx = \int \frac{2x-2}{x^2-2x+5}dx + \int \frac{dx}{x^2-2x+5} =$$

$$= \ln|x^2-2x+5| + \int \frac{dx}{(x-1)^2+4} = \ln|x^2-2x+5| + \frac{1}{4}\int \frac{2}{u^2+1}du =$$

$$= \ln|x^2 - 2x + 5| + \frac{1}{2}\arctan(\frac{x-1}{2}) + C.$$

(e)

$$\int_0^4 \log(1+x^2)\, dx = x\log(1+x^2)|_0^4 - \int_0^4 x \cdot \frac{2x}{x^2+1} dx =$$

$$= x\log(1+x^2)|_0^4 - 2\int_0^4 1 - \frac{1}{x^2+1} dx =$$

$$= x\log(1+x^2) - 2x + 2\arctan x)|_0^4 = 4\log(17) - 8 + 2\arctan(4).$$

[6] 4. Consider the solid whose base in the xy-plane is the region bounded by the curves $y = x^2$ and $y = 8 - x^2$, and whose cross-sections perpendicular to the x-axis are squares (with one side in the xy-plane). Find its volume.

Solution:

$$V = \int_{-2}^{2} (8 - x^2 - x^2)^2\, dx = (8x - \frac{2}{3}x^3)|_{-2}^{2} = \frac{2048}{15}$$

[8] 5. Determine whether each of the improper integrals is convergent or divergent, and if convergent find its value:

(a) $\int_0^\infty xe^{-x} dx$

Solution:

$$\int xe^{-x} dx = -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} + C.$$

So

$$\int_0^\infty xe^{-x} dx = \lim_{t\to\infty}(-xe^{-x} - e^{-x})|_0^t = \lim_{t\to\infty}(-te^{-t} - e^{-t} + 1) = \lim_{t\to\infty} \frac{-t-1}{e^t} + 1 = 1.$$

The integral converges.

(b) $\int_0^2 \frac{dx}{(x-1)^{2/3}}$

Solution: The integral is improper because the integrand is undefined at $x = 1$. Therefore

$$\int_0^2 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^2 \frac{dx}{(x-1)^{2/3}} =$$

$$= \lim_{t \to 1^-} \frac{(x-1)^{1/3}}{1/3}\Big|_0^t + \lim_{t \to 1^+} \frac{(x-1)^{1/3}}{1/3}\Big|_t^2 =$$

$$\lim_{t \to 1^-} 3(t-1)^{1/3} - 3(-1)^{1/3} + \lim_{t \to 1^+} 3(1)^{1/3} - 3(t-1)^{1/3} = 6.$$

The integral converges.

6 6. (a) Find a Riemann Sum S_n that approximates the integral

$$I = \int_0^1 \sqrt{x}dx$$

and such that $S_n > I$. Construct S_n using n intervals, where n is a positive integer.

Solution: We divide the interval $[0, 1]$ into n subintervals of equal length $\Delta x = 1/n$. We use the right-end point rule to build the Riemann sum $S_n = \Sigma_{i=1}^n \sqrt{\frac{i}{n}}\frac{1}{n}$. Since $\sqrt{x}$ is increasing on $[0, 1]$, it follows that $S_n > I$ as required.

(b) Prove that the sum of the square roots of the first n positive integers can be approximated as

$$\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} \approx \frac{2}{3}n^{3/2}$$

for large n.

Solution: From (a) we have $S_n = \Sigma_{i=1}^n \frac{\sqrt{i}}{\sqrt{n}}\frac{1}{n} = \frac{\Sigma_{i=1}^n \sqrt{i}}{n^{3/2}}$. For large n, $S_n \approx I = \frac{x^{3/2}}{3/2}\Big|_0^1 = \frac{2}{3}$, that is $\frac{\Sigma_{i=1}^n \sqrt{i}}{n^{3/2}} \approx \frac{2}{3}$ so $\Sigma_{i=1}^n \sqrt{i} \approx \frac{2}{3}n^{3/2}$.