8 1. Determine whether each of the following statements is true or false. **Provide justification**.

(a) True/False. $\frac{d}{dx}\int_0^{\pi/2} x^3\cos(t)\,dt = 0$.

Solution: The statement is true only if $x = 0$. If $x \neq 0$, $\frac{d}{dx}\int_0^{\pi/2} x^3\cos(t)\,dt = \frac{d}{dx}x^3\int_0^{\pi/2}\cos(t)\,dt = 3x^2\int_0^{\pi/2}\cos(t)\,dt \neq 0$. Thus, in general the statement is false.

(b) True/False. Let $F(x) = \int_1^{x^2} e^{-t^2}\,dt$. Then $F(x)$ is increasing for all x.

Solution: To determine whether F is increasing or decresing we need $F'(x)$. By the Fundamental Theorem of Calculus, after applying the Chain rule, we have $F'(x) = \frac{d}{dx}\int_1^{x^2} e^{-t^2}\,dt = e^{-(x^2)^2}(2x) = 2xe^{-x^4}$. We observe $F'(x) > 0$ only for $x > 0$. Thus the statement is false.

(c) True/False. $\int_{-\pi}^{\pi}|\sin(x)|\,dx = 2\int_0^{\pi}\sin(x)\,dx$.

Solution: The statement is true by symmetry of $|\sin(x)|$. In fact, $|\sin(x)|$ is periodic with period π, that is

$$\int_{-\pi}^{\pi}|\sin(x)|\,dx = \int_{-\pi}^{0}|\sin(x)|\,dx + \int_0^{\pi}|\sin(x)|\,dx = 2\int_0^{\pi}\sin(x)\,dx.$$

Alternatively,

$$\int_{-\pi}^{\pi}|\sin(x)|\,dx = \int_{-\pi}^{0}-\sin(x)\,dx + \int_0^{\pi}\sin(x)\,dx =$$

using the symmetry of the sine function, $\sin(-x) = -\sin(x)$, we have

$$\int_{-\pi}^{0}-\sin(x)\,dx = \int_{-\pi}^{0}\sin(-x)\,dx = \int_{\pi}^{0}-\sin(u)\,du = \int_0^{\pi}\sin(u)\,du$$

Thus,

$$\int_{-\pi}^{\pi}|\sin(x)|\,dx = \int_0^{\pi}\sin(u)\,du + + \int_0^{\pi}\sin(x)\,dx = 2\int_0^{\pi}\sin(x)\,dx.$$

(d) True/False If f is continuous for all x, then $\int_0^{\pi/2} f(\sin x)dx = \int_0^1 f(u)\cos u du$.

Solution: The statement is false. By making the substitution $u = \sin x$, we have $du = \cos x dx$ so $dx = \frac{du}{\sqrt{1-u^2}}$ and $\int_0^{\pi/2} f(\sin x)dx = \int_0^1 \frac{f(u)}{\sqrt{1-u^2}}$.

[6] 2. Consider the region R enclosed by the parabolas $y = -x^2 + x$ and $y = ax^2$, for $a > 0$. Find a value of a such that the area of R is $1/24$.

Solution: We need to find the points of intersection of the two parabola and then set up an integral representing the area between the parabolas.

The intersection is found by solving $ax^2 = -x^2 + x$ which yields $x = \frac{1}{a+1}$.

The area between the parabolas is given by

$$\int_0^{1/(a+1)} (-x^2 + x - ax^2)dx = (\frac{1}{2}x^2 - \frac{1}{3}(1+a)x^3)|_0^{1/(a+1)} = \frac{1}{6}\frac{1}{(a+1)^2},$$

thus

$$\frac{1}{6}\frac{1}{(a+1)^2} = \frac{1}{24}$$

which yields $(a+1)^2 = 4$ or $a = 1$.

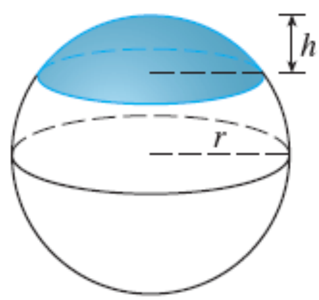
[20] 3. Evaluate any **four** integrals of your choice from the list below. Remember to include integration constants whenever appropriate. Continue your work on the next page if you need more space.

(a) $\int \frac{1}{x(x^2-1)}\, dx$ (b) $\int \sin^2(x)\cos^3(x)\, dx$ (c) $\int \sin(\ln(x))\, dx$

(d) $\int \frac{x}{x^2+2x+10}\, dx$ (e) $\int x^2 \ln(1+x^2)\, dx$

Solution:

[6] 4. Find the volume of the cap formed when a horizontal slice is made at a distance h from the top of a ball of radius r ($h < r$).


h
r

Solution: A horizontal cross section of the cap at a distance z from top is a disk of radius $\sqrt{r^2-(r-z)^2}$, so the cross-sectional area is $A(z)=\pi(\sqrt{r^2-(r-z)^2})^2 = \pi(2rz-z^2)$. It follows that the volume of the cap is given by

$$V=\int_0^h A(z)dz=\pi\int_0^h (2rz-z^2)dz=\pi(rz^2-\frac{1}{3}z^3)|_0^h=\pi(rh^2-\frac{1}{3}h^3).$$

8 5. Determine whether each of the improper integrals is convergent or divergent, and if convergent find its value:

(a) $\displaystyle\int_0^\infty xe^{-x^2}\,dx$

Solution:

$$\int_0^\infty xe^{-x^2}\,dx=\lim_{t\to\infty}\int_0^t xe^{-x^2}dx=\lim_{t\to\infty}-\frac{1}{2}e^{-x^2}|_0^t=\lim_{t\to\infty}\frac{1}{2}(1-e^{-t^2})=\frac{1}{2}$$

(b) $\displaystyle\int_0^2 \frac{dx}{x^2-1}$

Solution:

$$\int_0^2 \frac{dx}{x^2-1}=\int_0^1 \frac{dx}{x^2-1}+\int_1^2 \frac{dx}{x^2-1}.$$

Let's look at the first integral.

$$\int_0^1 \frac{dx}{x^2-1}=\lim_{t\to1^-}\int_0^t \frac{dx}{(x-1)(x+1)}=\lim_{t\to1^-}\int_0^t \frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx=$$

$$=\lim_{t\to1^-}\frac{1}{2}\ln|t-1|-\ln|t+1|-\frac{1}{2}\cdot 0=\lim_{t\to1^-}\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|=-\infty$$

(c) $\displaystyle\int_0^{\pi/4}\frac{\cos(x)}{x}dx.$

Solution: For $0 \leq x \leq \frac{\pi}{4}$, $\cos(x) \geq \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$. Then

$$\frac{\cos(x)}{x} \geq \frac{1}{\sqrt{x}}\frac{1}{x} > 0$$

so since $\int_0^{\pi/4} \frac{1}{x} dx$ diverges, so does $\int_0^{\pi/4} \frac{\cos(x)}{x} dx$.

6 6. Consider

$$\ln(x) = \int_1^x \frac{1}{t}\, dt.$$

(a) Use the definition of area as a definite integral to prove that $\frac{1}{2} < \ln(2) < 1$.

Solution: We observe that $\ln(2) = \int_1^2 \frac{1}{t}\, dt$, which is the area under the graph of $1/t$ for $1 \leq t \leq 2$. If we build a left Riemann sum S_{LH} with $\Delta x = 1$ (i.e., one subinterval), we have $S_{LH} = \frac{1}{1} \cdot 1 = 1$. Similarly, we can build a right Riemann sum S_{RH} with $\Delta x = 1$, $S_{RH} = \frac{1}{2} \cdot 1 = \frac{1}{2}$. Because $1/t$ is a decreasing function, any left Riemann sum gives an overestimate and any right Riemann sum gives an underestimate of the area. So $\frac{1}{2} < \ln(2) < 1$.

(b) Using the above definition of $\ln(x)$, prove that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln(n) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$

for $n > 0$.

Solution:

We observe that $\ln(n) = \int_1^n \frac{1}{t}\, dt$, so $ln(n)$ is the area under the graph of $1/t$ from 1 to n. If we build a left Riemann sum S_{LH} with $\Delta x = 1$ and $n - 1$ subintervals, we have

$$S_{LH} = \frac{1}{1} \cdot 1 + \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \cdots + \frac{1}{n-1} \cdot 1$$

If we build a right Riemann sum S_{RH} $\Delta x = 1$ and $n - 1$ subintervals, we have

$$R_{RH} = \frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1 + \cdots + \frac{1}{n} \cdot 1$$

Because $1/t$ is a decreasing function, we have

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln(n) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$