Science One Math

October 18, 2018

Today

- A few more examples of related rates problems
- A general discussion about mathematical modelling
- A simple growth model

Related Rates Problems

Problems where two or more variables are related to each other (the relationship is often derived from geometrical arguments) and we want to find the rate of change of one quantity given the rate of change of the other quantity.

Suppose gravel is added to the pile at a constant rate of k m^3/min . How fast is the height of the pile growing when the pile is h m high?

Assume the proportions of the pile remain constant as the pile grows and the pile if H m high and D m wide (at the base) when the conveyor belt stops. How fast is the height of the pile growing when the pile is h m high (with $h < H$) ?


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Solution: We are given $\frac{dV}{dt} = k$ m^3/min, we seek $\frac{dh}{dt}$.

Model $V = \frac{\pi}{3} r^2 h \Rightarrow \frac{dV}{dt} = \frac{\pi}{3}\left(2r\frac{dr}{dt}h + r^2\frac{dh}{dt}\right)$

Assume $\frac{h}{2r} = \frac{H}{D} \implies r = \frac{D}{2H} h \implies \frac{dr}{dt} = \frac{D}{2H}\frac{dh}{dt}$

hence

$$\frac{dV}{dt} = \frac{\pi}{4}\frac{D^2}{H^2}h^2\frac{dh}{dt} \implies \frac{dh}{dt} = \frac{4H^2}{\pi D^2}\frac{k}{h^2}$$

Solution strategy for related rates problems

- Draw a diagram, label quantities
- List time-dependent variables and constants
- Identify what variables are given
- Identify what variable you need to find to answer the question in the problem
- Write down a relationship between the relevant variables
- Use implicit differentiation (chain rule)
- If required, evaluate desired rate at the given instant.

Problem: Suppose a V shaped formation of birds forms a symmetric structure in which the distance from the leader to the last birds in the V is r and the distance between those trailing birds is D and the angle formed by the V is θ.

Suppose the shape is gradually changing: the trailing birds start to get closer so that their distance apart shrinks at a constant rate *k* while maintaining the same distance r from the leader. Assume that the structure is always in the shape of a V as the other birds adjust their positions to stay aligned in the flock.

At what rate is the angle θ changing? Express your answer in terms of r and D.

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Solution: At any instant $\frac{D}{2} = r\ \sin(\theta/2)$. We are given $\frac{dD}{dt} = -k$ (constant), r fixed.

$\Rightarrow \frac{dD}{dt} = r\ \cos\left(\frac{\theta}{2}\right)\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{-k}{r\cos(\theta/2)}$ need to express $\cos(\theta/2)$ in terms of r and D

At any instant $\cos(\theta/2) = \sqrt{1 - \sin^2(\theta/2)} = \sqrt{1 - D^2/(4r^2)}$.

$\Rightarrow \frac{d\theta}{dt} = \frac{-k}{r\sqrt{1-D^2/(4r^2)}} = \frac{-2k}{\sqrt{4r^2-D^2}}$

Observation: As the trailing birds get closer, $D \ll r \Rightarrow \frac{d\theta}{dt} = -\frac{k}{r}$ no longer depends on D

Problem: A lighthouse is L m from the straight shore on a rock. The light rotates at α rev/min. How fast is the light moving along the shore at a point X m from a point on the shore directly opposite the lighthouse?

Problem: A spider moves horizontally across the ground at a constant rate, *b*, pulling a thin silk thread with it. One end of the thread is tethered to a vertical wall at height *h* above ground and does not move. The other end moves with the spider. Find an expression for the rate of elongation of the thread in terms of *b* and *x*, the position of the spider at time t.

Problem: A lighthouse is L m from the straight shore on a rock. The light rotates at α rev/min. How fast is the light moving along the shore at a point X m from a point on the shore directly opposite the lighthouse?

Solution: $\frac{y}{L} = \tan \vartheta \quad \Rightarrow \quad \frac{dy}{dt} = L\,(\sec \vartheta)^2 \frac{d\vartheta}{dt}$

now evaluate when $y = X$ and $\frac{d\vartheta}{dt} = 2\pi\alpha \quad \Rightarrow \quad \frac{dy}{dt} = L(1 + \frac{X^2}{L^2})2\pi\alpha$

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Solution:

Let l be the length of the thread when the spider is at a distance x from the wall.

<u>Given</u>: x and l are functions of time,

h is a constant,

$dx/dt = b$, also constant.

<u>Need to find</u>: dl/dt when spider is at x.

Let's set up a relationship between x and l:

$l^2 = x^2 + h^2$, this is true at any instant.

Then, by chain rule and implicit differentiation,

$$2l\, dl/dt = 2\, x\, dx/dt + 0$$

$$dl/dt = (x/l)\, dx/dt$$

Given $dx/dt = b$, we get $dl/dt = (x/l)b$.

Finally, $l = (x^2 + h^2)^{1/2}$ therefore

$$dl/dt = (xb)/\,(x^2 + h^2)^{1/2}\ .$$

Other problems

1) *Making Coffee*: Coffee is draining from a conical filter into a cylindrical coffeepot at a rate of 10 cm^3/min. How fast is the level in the pot rising when the coffee is 5 cm deep? How fast is the level in the cone falling then?

2) *Walkers*: A and B are walking on straight streets that meet at right angles. A approaches the intersection at a speed a m/s; B moves away from the intersection at b m/s. At what rate is the distance between the walkers changing when A is 10 m from the intersection and B is 20 m from the intersection?

3) *A airplane problem*: An airplane climbing at an angle of 45^o passes over a ground radar station at an altitude of 8 km. At a later time the distance from the radar station to the airplane is 9 km and is increasing at the rate 700 km/h. Find the speed of the airplane at that time.

(*Hint*: draw a perpendicular from the radar station to the flight path of the plane)

Another problem: A sliding ladder

A 5-m ladder is leaning against a house when its base starts to slide away. By the time the base is 4 m from the house the base is moving at the rate of 2 m/s.

How fast is the top of the ladder sliding down the wall then?

At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?

At what rate is the angle θ between the ladder and the ground changing then?