Science One Math

October 23, 2018

Today

- A general discussion about mathematical modelling
- A simple growth model

Mathematical Modelling

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When assumptions are not met, the model may be a poor representation of reality $\Rightarrow$ model must be refined.

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Evolution equations are derived from basic principles.

Equations containing rates of change of unknown functions are called Ordinary Differential Equations (ODEs)

Which of the following equations has solution $y(t) = 3e^{2t}$?

A. $\frac{dy}{dt} = 3y$

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To check whether a function y is a solution to a differential equation, compute y' and plug into equation.

$\boldsymbol{\frac{dy}{dt} = ky}$ for any constant k ☞ a simple ODE with **solution** $\boldsymbol{y = Ce^{kt}}$

Note: $C = y(0)$

Exponential Models

a few examples

A simple model for uncontrolled growth

$P(t)$ = # of individuals in population at time t,

b = per capita birth rate (per unit time)

m = per capita mortality rate (per unit time).

Assumption:

b & m both constant over time

Goal: Given the initial size of the population, find a function $P(t)$ that predicts the population size at time t.

Approach:

Change in population over a time interval Δt is ΔP:

$\Delta P = P(t + \Delta t) - P(t) =$ [# of births] – [# of deaths] **balance equation**

(basic principle)

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$$P(t+\Delta t) - P(t) = b\,P\,\Delta t\ - m\,P\,\Delta t \quad \Rightarrow \quad \frac{P(t+\Delta t) - P(t)}{\Delta t} = (b-m)P$$

Δt→0, $\frac{dP}{dt} = (b-m)P$

Let $r = b - m$, **net per capita growth rate** (per unit time): $\boxed{\boldsymbol{\frac{dP}{dt} = rP}}$

(intrinsic parameter)

Malthusian Model (or exponential growth)

Uncontrolled growth model: $\boldsymbol{\frac{dP}{dt} = rP}$

Solution (guess and check): $P(t) = P(0)e^{rt}$

Simplistic model, reasonable for population with unlimited resources (food, land, etc.)

(Thomas Malthus, “An Essay on the Principle of Population”, 1789)

A population of animals has a per-capita birth rate of $b = 0.08$ per year and a per-capita death rate of $m = 0.01$ per year. Assuming b and m are constant, the population size $P(t)$ is found to satisfy which one of the following equations?

A. $\frac{dy}{dt} = 7y$

B. $\frac{dy}{dt} = 0.07y$

C. $\frac{dy}{dt} = 0.09y$

D. $P(t) = P(0)e^{7t}$

E. $P(t) = P(0)e^{0.07t}$

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Exponential Growth Models

$\frac{dP}{dt} = rP$ has solution $P(t) = P(0)e^{rt}$ for some constant *r*

- $P \geq 0$ *provided* $P(0) \geq 0$
- if $\boldsymbol{r > 0} \Rightarrow$ **growth** (birth rate > mortality rate)
- if $\boldsymbol{r < 0} \Rightarrow$ **decay** (birth rate < mortality rate)

Two main properties:

1. Fixed relative growth rate
2. Fixed doubling time (or half life)

Property 1: Fixed relative growth rate

$\dfrac{\boldsymbol{dP}}{\boldsymbol{dt}}$ is the **growth rate** (per unit time) changes over time

depends on the size of population

$\dfrac{\frac{dP}{dt}}{P} = r =$ **% growth rate** (per unit time) constant over time

(relative to the size of population)

Property 2: Fixed Doubling Time

Suppose $P(t) = P(0)e^{rt}$ given some initial condition $P(0) = P_0$.

Let τ = time it takes P to double in value (**doubling time**)

Does τ depend either on time or population size?

$$P(\tau) = 2P(0) = 2P_0$$

$$P(0)e^{r\tau} = P_0 e^{r\tau} = 2P_0$$

$$e^{r\tau} = 2$$

$$r\tau = \ln 2$$

(Doubling time) $\boldsymbol{\tau = \frac{\ln 2}{r}}$ *constant for a specific population*

Example: World population

Data: 2 billion in 1927, 4 billion in 1974

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Solution:

Assumption: exponential growth $\frac{dP}{dt} = rP,\ r$ is constant over time,

t in years since 1927, *P* in billions.

1927: $t = 0, P(0) = 2$

1974: $t = 1974 - 1927 = 47, P(47) = 4$ doubled!

$r = \frac{\ln 2}{47} = 0.0147\ (\sim 1.5\%)$, our model is $P(t) = P(0)e^{[(ln2)/47]t}$

Predictions: 1999, $t = 1999 - 1927 = 72,\ P(72) = 2e^{\frac{72(\ln 2)}{47}} \sim 5.78$

2016, $t = 2016 - 1927 = 89,\ P(89) = 2e^{\frac{89(\ln 2)}{47}} \sim 7.43$

Bacteria population

Assume that the number of bacteria at time t (in minutes) grows exponentially. Suppose the count in the bacteria culture was 300 cells after 15 minutes and 1800 after 40 minutes.

What was the initial size of the culture?

When will the culture contain 3000 bacteria?

Modelling rates of decay: Radioactivity

$N(t) =$ # of atoms in sample at time t

k = probability of decay (per unit time) ***constant*** *for a given chem. element*

Change in amount of material over a time interval Δt

$$\Delta N = N(t + \Delta t) - N(t) = -k\ N\ \Delta t \ \Rightarrow \quad \frac{N(t+\Delta t)-N(t)}{\Delta t} = -k\ N$$

$\Delta t \ \rightarrow 0 \qquad \boldsymbol{\frac{dN}{dt} = -kN}$ where *k* is a constant

In 1986 the Chernobyl nuclear power plant exploded, and scattered radioactive material over Europe. Of particular note were the two radioactive elements: **iodine-131 (half-life 8 days)** and cesium-137 (half-life 30 years).

Which of the following differential equations can be used to predict how much of iodine-131 would remain over time?
Let $y(t)$ be the amount of iodine-131 at time t.

A. $y' = -0.023\,y$
B. $y' = -0.087\,y$
C. $y' = -8.00\,y$
D. $y' = -11.54\,y$
E. $y' = -30.0\,y$

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A solution of the differential equation $y' = -\, k\, y$ where k = 0.0866 is

$y = y_0\, e^{-0.0866\, t}$ iodine-131 at time t

What are the units of the constant k = 0.0866?

A. days
B. 1/days
C. years
D. 1/years
E. k is a dimensionless number

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How long it would take for iodine-131 to decay to 0.1% of its initial level?

$y = y_0\, e^{-0.0866\, t}$ (where t is measured in days)

A. About 80 days
B. About 53 days
C. About 27 days
D. It depends on the initial level of iodine-131

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Suppose that 1000 rabbits are introduced onto an island where they have no predators. During the next five years the rabbit population grows exponentially. After the first 2 years that population grew to 3500 rabbits. After the first 5 years a rabbit virus is sprayed on the island and after that the rabbit population decays exponentially. 2 years after the virus was introduced, the population dropped to 3000 rabbits. How many rabbits will there be on the island 10 years after they were introduced?

Suppose an initial dose of 100 mg of a drug is administered to a patient. Assume it takes 36 hr for the body to eliminate half of the initial dose from the blood stream.

Find an exponential function that models the amount of drug in the blood at t hr. (your answer should have no unknown constants)

At what rate (in mg per hr) is the drug eliminated from the blood stream after 36 hr since the initial dose?

If a second 100-mg dose is given 36 hr after the first dose, how much time is required for the drug-level to reach 1mg?

Log-linear models

$P(t) \;=\; P(0)\,e^{rt}$ an exponential model can be converted to a linear model using logs:

$$\ln P \;=\; \ln(P(0)\,e^{rt})$$

$$\ln P \;=\; \ln(P(0)) \;+\; \ln(e^{rt})$$

$$\ln P \;=\; \ln(P(0)) \;+\; r\,t$$

$$y \;=\; A \;+\; r\,t$$