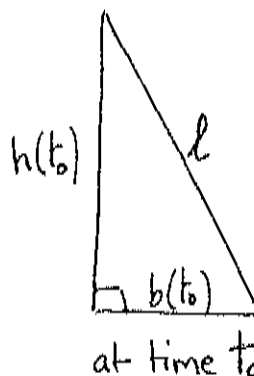
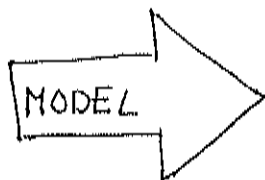
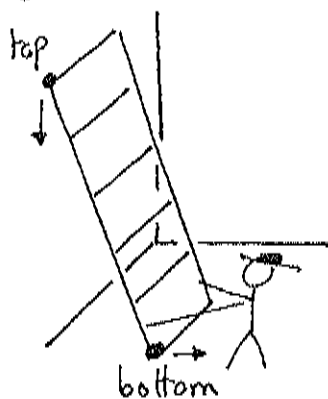


The moving ladder

During our Tuesday class, we discussed how to model the situation of a ladder leaning against a wall using a right triangle. Since the ladder is sliding, we need to take into account time as well, so we have:



Where

- $b(t)$ = the function describing the distance between the wall and the bottom of the ladder at time t
- $h(t)$ = the function describing the distance between the ground and the top of the ladder at time t
- l = the length of the ladder

Recall how we discussed the different role played by all the various letters involved here: b and h are actually functions of the time, which we represent with the variable t and l is a constant which is independent from time.

Once variables have been correctly and precisely identified, we seek to understand how they relate. This worksheet will guide you through the process of reading three other situations and their models. You will be able to identify correctly the various variables at play and become familiar with three other situations we expect you to know well enough to be able to recreate them and adapt them if necessary (this will be practiced in our Thursday class).

The way in which the different variables are related in the moving ladder situation is given by:

For now, think about it, complete the rest of this worksheet, you should be able to answer this question by the end of it.

The melting snowball

Consider a snowball which is melting under the sun. There are two interesting rates at play in this situation: the rate at which the volume of the snowball is changing and the rate at which the radius of the snowball is changing. Clearly those two rates are related, but can we be more precise?

Usually, you would see a word problem of the following flavour:

A snowball is melting under the sun. At some precise instant, we take two measurements and discover that its volume is 40 cubic centimetres and its volume is changing at the rate of 10 cubic centimetres per minute. At what rate is the radius of the snowball changing at that moment?

The model

To model this situation, we will consider that the snowball is a perfect sphere of radius r and of volume V . Each of these two variables actually change with time, so to be more precise, each actually are function of time which we will represent with the variable t . We denote by t_0 the specific moment at which the situation is considered.



The modelled snowball at time t_0

Finding a relationship

It is well known to us that the volume of a sphere is in relationship to its radius by the formula:

$$V = \frac{4}{3}\pi r^3$$

And to be more precise since each of these variables changes with time, we can write:

$$V(t) = \frac{4}{3}\pi(r(t))^3$$

In other words, we now have an equation which related the two functions we are interested in (the functions describing the volume and the radius over time). Now it feels like calculus might help us! (More on this on Thursday of course).

Questions

1. Using this model, how do we represent the information given in the word problem by the phrase "its volume is 40 cubic centimetres"?
2. Is it possible to compute $r(t_0)$ from the information given in the word problem? If so how, if not what would we need?

Ships moving along perpendiculars

Consider two ships moving along paths that are perpendicular (see the picture below). A usual question is to know at what rate the distance between them changes if we know their respective position and speed.

A typical word problem would be:

Two boats leave a port at the same time, one traveling west and the other traveling south. At the moment where the first one is 15 kilometres away from the port with a speed of 20 kilometres per hour, the other boat is 20 kilometres away and has a speed of 30 kilometres per hour. At what rate is the distance between the two boats changing?

The model

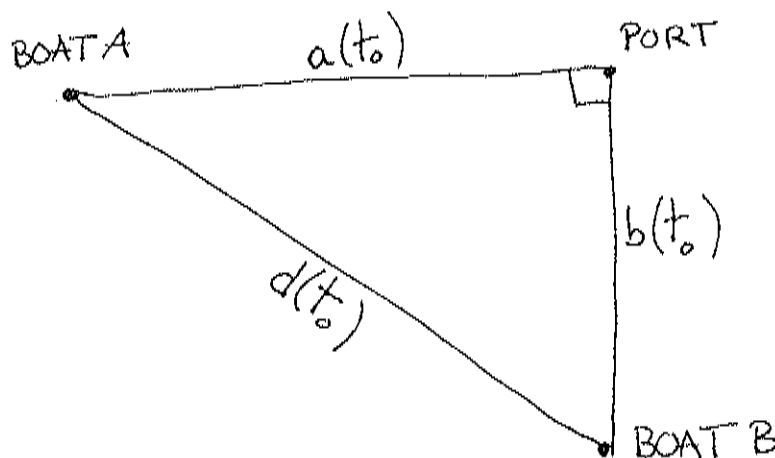
To model this situation, we draw a triangle whose three corners are the port and the two boats. Let us label the first ship by A and the second by B. Contrary to the previous situation, there is no clear notation for any of the variables at play, so we create our own. Let us define

- a = the distance between the port and boat A
- b = the distance between the port and boat B
- d = the distance between boats A and B

As previously, some of those variables actually change with time (which is always a hidden variable at first) and so to be more precise, we can write that

- $a(t)$ = the function giving the distance between the port and boat A at time t
- $b(t)$ = the function giving the distance between the port and boat B at time t
- $d(t)$ = the function giving the distance between boats A and B at time t

The specific moment at which we look at the position and speed of the boats is specific value of time; will denote that instant by t_0 . Now we can draw the following diagram of the modeled situation:



The modelled situation at time t_0

Finding a relationship

Since the paths of the boats are perpendicular, the triangle that we are considering is a right triangle which allows us to use the Pythagorean theorem to describe how those variables are related:

$$a^2 + b^2 = d^2$$

Or more precisely since those variables are all functions of time:

$$(a(t))^2 + (b(t))^2 = (d(t))^2$$

This equations relates the three functions that we are interested in and we will discuss on Thursday how to use calculus to answer the initial question of the word problem.

Questions

1. Using this model to discuss the given word problem, what can you say about the following objects:

$a(t)$, $a(t_0)$, $a'(t)$, $a'(t_0)$, $b(t)$, $b(t_0)$, $b'(t)$, $b'(t_0)$, $d(t)$, $d(t_0)$, $d'(t)$, $d'(t_0)$

Describe each then explain which one are given in the statement of the word problem, which ones can be computed and how, and finally which one corresponds to the answer of the problem.

2. Explain why this model only works if the ships move along perpendicular directions. How could we model the situation if that is not the case (for example, if they were heading in opposite directions like north and south, or at some other angle to each other)?

A draining conical tank

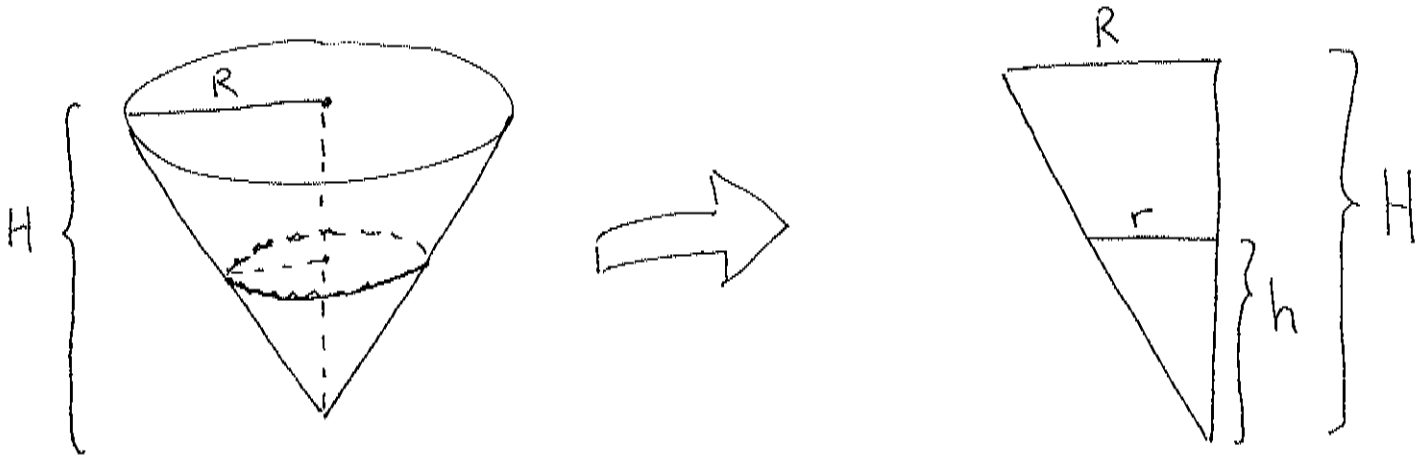
The last situation we will discuss here is the case of a draining conical tank. Imagine a tank containing water which has the shape of an upside-down cone (nose at the bottom as in the picture below). If there is a hole at its bottom to drain it, we would like to know how the rate at which its volume is changing is related to the speed at which the water level is moving.

Typically, you would see a question such as:

Consider a tank of water that is draining by a hole at its bottom. The tank has the shape of an inverted cone that is 5 metres high and that has a base radius of 3 metres. At the precise moment at which the water level is 2 metres, we measure that the tank is draining at a rate of 0.7 cubic metres per second. At that instant, what is the speed at which the water level is decreasing?

The model

Drawing a diagram of this situation should make you realize quickly that there are actually two cones at play! The tank and the body of water. Drawing a vertical cut of the situation makes things easier to look at:



This means that we will require several variables to describe the situation. Let us define the following variables.

- R = the radius of the tank
- H = the height of the tank
- V = the volume of the tank
- r = the radius of the body of water
- h = the height of the body of water
- v = the volume of the body of water

As before, we need to think about the precise identity of each of these variables. Since the shape of the tank is not changing over time, the variables R , H and V actually are constants. On the other hand, the body of water is changing since the tank is draining and hence the variables r , h and v are functions of time (denote once again by the variable t). And as previously we will denote by t_0 the mysterious moment at which the situation is considered. So more precisely we can rewrite things as:

- R = the radius of the tank
- H = the height of the tank
- V = the volume of the tank
- $r(t)$ = the function describing the radius of the body of water at some time t
- $h(t)$ = the function describing the height of the body of water at some time t
- $v(t)$ = the function describing the volume of the body of water at some time t

Finding a relationship

Trying to relate those variables is fairly obvious now since we know very well how to compute the volume of a cone. And since we have two cones at play, we obtain two equations:

$$V = \frac{1}{3}\pi R^2 H$$

and

$$v(t) = \frac{1}{3}\pi (r(t))^2 h(t)$$

Now we are not totally done since initially we asked how to relate the volume of the body of water to its height and it doesn't feel like it should depend of its radius though it looks as if from the above equation. What can we do about this? Well the two cones (the tank and the body of water) are clearly similar (they have the same shape) hence we can use similarities of triangles to claim that

$$\frac{R}{H} = \frac{r}{h}$$

And from this we can derive an equation which will allow us to get rid of the variable r in our main equation (at the cost of more little letters everywhere!):

$$r = \frac{R}{H}h \quad \text{or more precisely} \quad r(t) = \frac{R}{H}h(t)$$

And so we can substitute this back into the equation that relates $v(t)$ and $h(t)$ to obtain a new equation without a $r(t)$ but instead more $h(t)$ and R and H (which we know or are at least interested in):

$$v(t) = \frac{1}{3}\pi \left(\frac{R}{H}h(t)\right)^2 h(t)$$

or if we do some straightforward algebra (which you should try yourself)

$$v(t) = \frac{R^2}{3H^2}\pi (h(t))^3$$

Questions

1. What makes this model more or less complicated than some others?
2. What has improved in the last line ($v(t) = \dots$) compared to the original version of the model for $v(t)$ above?
3. Go back to the ladder model and answer the question! **Then go to VISTA and enter your answers!**

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