Learning Goals for MATH 220: "Mathematical Proof"

The following set of learning goals have been developed largely from past midterm and final exams for this course, and are intended as an aid to your studying. While they cannot list everything you could be asked to do on an exam, these goals are representative of the topics and types of questions that have been on exams in the past. We therefore recommend that you use these as a guideline and that in your studying you search out problems that represent each learning goal and practice (or even better, test yourself) on these problems.

The precise content of the course will depend on time and instructor preferences. Please ask your instructor if you have any questions about the topics covered.

Course-Level Goals

1. Describe what a mathematical proof is and why proofs are important in mathematics.

2. Communicate mathematics, including definitions, theorems and proofs in a clear and precise way. Employ the notation, format and completeness conventions of the mathematics community.

3. Recognize the different types of proof and outline the logical structure of each.

4. Given a proof, identify the assumptions and conclusion, any lemmas used, the method of proof and the main logical steps of the proof.

5. Given a statement about numbers, sets, functions, cardinality, suprema or infima, sequences or series, select an appropriate method of proof, sketch a proof and use it to write a clear and complete proof.

6. Construct statements about numbers, sets, functions, cardinality, suprema or infima, sequences or series, determine their truth value, and prove or disprove them.

Topic-Level Goals

1. Communicating Mathematics

   - State definitions, properties and theorems in a precise and clear manner.
   - Construct arguments that are precise and clear, and include sufficient detail for the reader to understand.
   - Recognize the difference between the formal write up of a proof, and the process of constructing a proof.
   - Transform the "scrap work" of proof construction (proof outline and computations) into a formal proof that is neat, legible and follows a logical order.

2. Sets

- Recall and state set definitions, including: set, members/elements of a set, empty set, subset, equality of sets, proper subset, power set, cardinality of a set, union and intersection of sets, union and intersection of a collection of indexed sets, disjoint sets, relative complement/difference of sets, complement of a set, set partition, ordered pair, Cartesian product of sets.

- Interpret and use set notation, including: such that $s.t.$, $|$, $:$, an element of $\in$, set definition $\{x \in \mathbb{R} \ s.t. \ P(x)\}$, empty set $\emptyset$, universal set U, subsets $A \subset B$, $A \supset B$, $A \subseteq B$, $A \supseteq B$, power set $\mathcal{P}(A)$, cardinality of a set $|A|$, union of sets $A \cup B$, intersection of sets $A \cap B$, union and intersection of an indexed collection of sets $\bigcup_{\alpha \in I} S_\alpha$, $\bigcap_{\alpha \in I} S_\alpha$, relative complement/difference of sets $A - B$, $A \backslash B$, complement of a set A^c, $\bar{A}$, Cartesian product of sets $A \times B$, commonly used sets of numbers $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{I}$, $\mathbb{C}$, open, closed, half-closed and unbounded intevals of $\mathbb{R}$ (a, b), $[a, b]$, $(a, b]$, $[a, b)$, (a, ∞), $(-\infty, b)$, $(-\infty, \infty)$.

- Compute the power set of a finite set.

- Compute the cardinality of a finite set.

- Recall and apply the rule for the cardinality of power sets.

- Generate examples of sets and power sets with given cardinality.

- Compute the union, intersection and complement of sets or indexed collections of sets.

- Compute the relative complement of one set in another.

- Recall and state De Morgan's laws for sets.

- Draw Venn diagrams to illustrate set operations and relationships between sets.

- Construct a partition of a set.

3. Logic

- Recall and state logic definitions, including: disjunction of two statements, conjunction of two statements, implication/conditional statements, hypothesis of an implication, conclusion of an implication, converse of an implication, contrapositive of an implication, biconditional/equivalent statements, tautology, contradiction.

- Interpret and use logic notation, including: negation of a statement: $!P$, $\sim P$, $\neg P$, conjunction ("and") $P \wedge Q$, disjunction ("or") $P \vee Q$, implication/conditional $P \Rightarrow Q$, $P \Leftarrow Q$, biconditional/equivalence $P \Leftrightarrow Q$, P iff Q, logical equivalence $P \equiv Q$, universal quantifier $\forall$, existential quantifier $\exists$, open sentence $P(x)$.

- Differentiate between statements and open sentences.

- Construct and use truth tables to determine the truth values of statements.

- Construct and determine the truth value of statements that include quantifiers and logical connectives.

- Convert biconditional statements into compound implication statements and vice versa.

- Determine, with justification, the truth value of statements (about subsets of $\mathbb{R}$) with nested quantifiers.

- Negate plain English implication statements.

- Negate statements including logical connectives, quantifiers, sets and inequalities.

- Determine and write the converse and contrapositive of plain English implication statements.
- Determine and write the converse and contrapositive of statements including logical connectives, quantifiers, sets and inequalities.
- State and apply the following logic laws to convert compound statements into logically equivalent statements: the commutative laws, the associative laws, the distributive laws, De Morgan's laws.

4. Direct Proof and Proof by Contrapositive

- Recall and state the following formal definitions: odd and even integers, "a divides b," for integers a and b, congruence modulo n, for integer n.
- Interpret and use the following notation: divisibility $a \mid b$, congruence $a \equiv b \bmod n$.
- Be familiar with the following terms and how they are used in mathematical writing: axiom, theorem, corollary, lemma.
- Identify, and construct when asked, trivial and vacuous proofs.
- Recognize and follow the logical progress of direct proofs, including proof by cases.
- Recall and use the result that an implication is logically equivalent to its contrapositive.
- Determine when to use direct proof and proof by contrapositive to prove statements about the parity of integers, divisibility, congruences and explicit sets.
- Use direct proof and proof by contrapositive to prove statements about the parity of integers, divisibility, congruences and explicit sets.
- Disprove by construction of counterexamples that a statement about the parity of integers, divisibility, congruences or explicit sets is false.
- Apply set definitions and logical operators to convert statements about sets to logically equivalent statements.
- Prove or disprove that a given set is a subset of another given set.
- Prove or disprove that two given sets are equal.
- Recall, prove and apply the commutative, associative, distributive and De Morgan's laws for sets.

5. Equivalence Relations (optional topic - coverage will depend on available time)

- Recall and state equivalence relation definitions, including: ordered pair, Cartesian product, relation, equivalence relation, equivalence class, partition.
- Interpret and use equivalence relation notation, including: ordered pair (a, b), Cartesian product $A \times B$, relation aRb, equivalence class E_x, $[x]$, partition $\mathscr{P}$.
- Recall and state the properties that an equivalence relation must possess.
- Given a relation, determine if it is an equivalence relation by proving or disproving whether required properties (reflexivity, symmetry and transitivity) are satisfied.
- Construct equivalence relations on a given set.
- Recall the relationship between equivalence relations and partitions of a set and use this to construct a partition from an equivalence relation and vice versa.

6. Functions

- Recall and state function definitions, including: relation, function, domain, codomain, range, image of a set under a function, preimage of a set under a function, composition of two functions, injective/one-to-one function, surjective/onto function, bijective function, identity function, inverse function.
- Interpret and apply function notation, including: $f : A \rightarrow B$, range $\mathrm{rng}f$, composition of functions $f \circ g$, $f(g(x))$, identity function i_A, preimage of a set $f^{-1}(D)$, function inverse f^{-1}.
- Apply function definitions to convert statements about functions to logically equivalent statements.
- Determine whether statements about images and preimages of sets are true or false and prove or disprove them.
- Determine conditions under which a given function is injective, surjective or bijective.
- Generate examples of injective, surjective and bijective functions, as well as non-injective and non-surjective functions.
- Prove that a given function satisfies the definition of injective, surjective or bijective.
- Apply the definition of a surjective or injective function to prove or disprove that related functions are surjective or injective.
- Recall and use the relationship between the cardinalities of two finite sets and the existence of injections, surjections and bijections.

7. Proof by Contradiction

- Determine when to use proof by construction and disproof by counterexample.
- Recall and state the "law of the excluded middle" and explain how it relates to proof by contradiction.
- Describe the logical steps for the method of proof by contradiction.
- Recognize proof by contradiction and contexts in which it is a useful approach.
- Prove by contradiction simple statements about numbers.
- Prove by contradiction that numbers of a given form are irrational.

8. Cardinalities of Sets

- Recall and state cardinality definitions, including: finite set, infinite set, equal cardinality/numerically equivalent/equinumerous sets, lesser or greater cardinality, countable set, uncountable set, denumerable set.
- Interpret and use cardinality notation, including: cardinality of a set $|A|$, comparing cardinalities of sets $>, <, =, \geq, \leq$, cardinal number of a denumerable set $\aleph_0$, continuum cardinality c.
- Generate examples of sets with different cardinalities.
- Compute the cardinality of given sets.
- Recall and apply the symmetric and transitive properties of cardinalities of sets to prove statements about cardinalities of sets.

- Recall and use the relationship between the cardinalities of two sets and the existence of injections, surjections and bijections.
- Determine whether statements about the existence of injections, surjections or bijections between sets are true or false and prove or disprove them.
- Prove $\geq$, $\leq$ or $=$ for two sets by construction of injections, surjections and bijections between the two sets.
- Apply known cardinality relationships ($>, <, =, \geq$ or $\leq$) between sets to prove statements about the cardinalities of related sets.
- Determine whether a given set of numbers is countable.
- Recognize that there is a difference between countable and uncountable infinite sets and provide examples of each.

9. Mathematical Induction

- Recall and state the following definitions: least element/minimum/smallest element of a set, a well-ordered subset of $\mathbb{R}$.
- Interpret and use notation for summation $\displaystyle\sum_{j=1}^{n} a_j$.
- Recall and state the Well-Ordering principle of $\mathbb{N}$.
- Determine whether given subsets of the reals have a least element or are well-ordered.
- Generate examples of sets that are well-ordered, have a least element, are not well-ordered, and do not have a least element.
- Recall and state the principle of mathematical induction.
- Apply mathematical induction to prove statements about inequalities, summation and divisibility.
- Adapt mathematical induction to base cases other than $n = 1$.
- Write well-structured and clear mathematical induction proofs.

10. Supremum and Infimum

- Recall and state supremum and infimum definitions, including: supremum, infimum, maximum, minimum, upper bound, lower bound, bounded subset of $\mathbb{R}$.
- Interpret and use notation for supremum and infimum: supremum sup S, infimum inf S.
- Recall and state the completeness axiom of $\mathbb{R}$.
- Recall and state the Archimedean property of $\mathbb{N}$ in $\mathbb{R}$.
- Determine the existence of suprema, infima, maxima and minima for a given subset of the real numbers.
- Compute the supremum, infimum, maximum and minimum (when they exist) of a given subset of $\mathbb{R}$.
- Prove the existence of infimum or supremum for a given set of real numbers.
- Apply the existence of infimum or supremum for a given set to prove the existence of upper bounds, lower bounds, infimum or supremum for related sets.
- Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$ and explain what this means.

11. Proofs in Calculus

Sequences:

- Recall and state sequence definitions, including: sequence, convergent sequence, divergent sequence, divergence of a sequence to $\pm\infty$, bounded sequence, monotone sequence, ceiling of a real number, floor of a real number.
- Interpret and use notation for sequences, including: sequence definition ex. $\{s_n\}$, $s_n = \frac{1}{n}$, sequence convergence $a_n \to a$, limit of a sequence $\lim_{n\to\infty} a_n$, ceiling $\lceil r \rceil$, floor $\lfloor r \rfloor$.
- Recall and apply the triangle inequality.
- Generate a chain of inequalities to prove that a desired inequality is satisfied.
- Visualize convergent and divergent sequences.
- Determine if a given sequence converges or diverges.
- Recall and apply limit laws to determine (but not to prove) the limit of a given sequence.
- Generate multiple examples of sequences that converge to a given value.
- Prove from the definition of convergence that a given sequence converges.
- Apply the definition of a convergent sequence to prove related statements about the sequence.

Series (coverage will depend on available time):

- Recall and state series definitions, including: series/infinite series, n^{th} partial sum of a series, convergent series, divergent series, divergence of a series to $\pm\infty$, absolutely convergent series, conditionally convergent series.
- Interpret and use notation for series, including: series $\sum_{i=1}^{\infty} a_i$, divergence of a series to $\pm\infty$ $\sum_{n=1}^{\infty} a_n = \pm\infty$.
- Prove from the definition that a given series converges.
- Prove that a given series diverges to $\pm\infty$.
- Recognize a geometric series, determine if it is convergent and (when convergent) compute its sum.
- Recognize and simplify telescoping series.
- Recall and apply the theorem for addition and scalar multiplication of convergent series.
- Determine when to use, and apply where appropriate: the comparison test, the ratio test, and the integral test for series.

Limits of Functions (coverage will depend on available time):

- Recall and state limit definitions, including: deleted neighbourhood, limit of a function.
- Interpret and use notation for limits of functions: $\lim_{x\to a} f(x)$.
- Determine the limit of a function.
- Prove, from the definition, the limit of a given function, including polynomials and ratios of polynomials.
- Prove that the limit of a given function does not exist.