Solutions to
Lines and Planes
Practice Problems

Find the parametric form $\vec{\mathbf{x}}(t) = \vec{\mathbf{p}} + t\vec{\mathbf{d}}$ of a line given the following information:

1. Two points $\vec{\mathbf{p}}_1$ and $\vec{\mathbf{p}}_2$ on the line, say $\vec{\mathbf{p}}_1 = [1, -1, 3]$ and $\vec{\mathbf{p}}_2 = [2, 3, -1]$:

   Answer

   Let $\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 = [1, 4, -4]$ be the direction vector $\vec{\mathbf{d}}$ and let $\vec{\mathbf{p}}_1$ be a point on the line, then the line can be represented by $\vec{\mathbf{x}}(t) = [1, -1, 3] + t[1, 4, -4]$.

2. One point $\vec{\mathbf{p}}$ on the line and a direction vector $\vec{\mathbf{d}}$, say $\vec{\mathbf{p}} = [1, 2, 3]$ and $\vec{\mathbf{d}} = [6, 5, 4]$.

   Answer

   $\vec{\mathbf{x}}(t) = [1, 2, 3] + t[6, 5, 4]$.

3. One point $\vec{\mathbf{p}}$ on the line and parallel ($\|$) to another line $\vec{\mathbf{a}} + t\vec{\mathbf{b}}$, say $\vec{\mathbf{p}} = [1, 0, 2]$, $\vec{\mathbf{a}} = [2, 2, -2]$, and $\vec{\mathbf{b}} = [1, 4, 5]$.

   Answer

   Let the direction vector be $\vec{\mathbf{b}} = [1, 4, 5]$ and $\vec{\mathbf{p}} = [1, 0, 2]$, thus $\vec{\mathbf{x}}(t) = [1, 0, 2] + t[1, 4, 5]$.

4. One point $\vec{\mathbf{p}}$ on the line and $\perp$ to another line $\vec{\mathbf{a}} + t\vec{\mathbf{b}}$, say $\vec{\mathbf{p}} = [1, 0, 2]$, $\vec{\mathbf{a}} = [2, 2, -2]$, and $\vec{\mathbf{b}} = [1, 4, 5]$.

   Answer

   Let the direction vector be any nonzero vector $\vec{\mathbf{d}} = [d_1, d_2, d_3]$ such than $\vec{\mathbf{d}} \cdot \vec{\mathbf{b}} = 0$, i.e. $d_1 + 4d_2 + 5d_3 = 0$. The choice $\vec{\mathbf{d}} = [1, 1, -1]$ works so $\vec{\mathbf{x}}(t) = [1, 0, 2] + t[1, 1, -1]$. Alternatively, $\vec{\mathbf{d}}$ could be taken to be $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$, where $\vec{\mathbf{c}}$ is any vector not $\|$ to $\vec{\mathbf{b}}$.

5. One point $\vec{\mathbf{p}}$ on the line and $\perp$ to a plane $ax + by + cz + d = 0$, say $\vec{\mathbf{p}} = [1, 0, 2]$, $a = 2$, $b = -4$, $c = 5$, and $d = 1$.

   Answer

   Let the direction vector be $\vec{\mathbf{b}} = [a, b, c] = [2, -4, 5]$, then $\vec{\mathbf{x}}(t) = [1, 0, 2] + t[2, -4, 5]$.

6. One point $\vec{\mathbf{p}}$ on the line and $\|$ to a plane $ax + by + cz + d = 0$, say $\vec{\mathbf{p}} = [1, 0, 2]$, $a = 2$, $b = -4$, $c = 5$, and $d = 1$.

   Answer

   Let $\vec{\mathbf{b}} = [a, b, c]$ be the normal vector to the plane, and let the direction vector of the line to be any nonzero vector $\vec{\mathbf{d}} = [d_1, d_2, d_3]$ such than $\vec{\mathbf{d}} \cdot \vec{\mathbf{b}} = 0$, i.e. $2d_1 - 4d_2 + 5d_3 = 0$. The choice $\vec{\mathbf{d}} = [1, -2, -2]$ works so $\vec{\mathbf{x}}(t) = [1, 0, 2] + t[1, -2, -2]$. Alternatively, $\vec{\mathbf{d}}$ could be taken to be $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$, where $\vec{\mathbf{c}}$ is any vector not $\|$ to $\vec{\mathbf{b}}$.

7. The line is the intersection of the two planes: $a_1x+b_1y+c_1z+d_1 = 0$ and $a_2x+b_2y+c_2z+d_2 = 0$, where $a_1 = 2$, $b_1 = -4$, $c_1 = 5$, $d_1 = 1$, $a_2 = 1$, $b_2 = -7$, $c_2 = 1$, and $d_2 = 4$.

Answer

Let the direction vector be $\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2 = [2, -4, 5] \times [1, -7, 1] = [31, 3, -10]$. The point $\vec{\mathbf{p}}$ must lie on both planes, therefore $\vec{\mathbf{p}} = [x, y, z]$ must satisfy both the equations:

$$\begin{array}{ll} (1) & 2x - 4y + 5z + 1 = 0 \\ (2) & x - 7y + z + 4 = 0 \end{array}$$

This system has infinitely many solutions, but we just need any one for our point $\vec{\mathbf{p}}$. $(1) - 2(2) \Rightarrow 10y + 3z - 7 = 0$, so take $y = 1$, $z = -1$ and substitute this into (2) to get $x = 4$. Thus $\vec{\mathbf{x}}(t) = [4, 1, -1] + t[31, 3, -10]$.

8. One point $\vec{\mathbf{p}}$ on the line and $\perp$ to each of two given lines: $\vec{\mathbf{a}}_1 + t\vec{\mathbf{b}}_1$ and $\vec{\mathbf{a}}_2 + t\vec{\mathbf{b}}_2$, where $\vec{\mathbf{p}} = [1, 0, 2]$, $\vec{\mathbf{a}}_1 = [2, 2, -2]$, $\vec{\mathbf{b}}_1 = [1, 4, 5]$, $\vec{\mathbf{a}}_2 = [2, 4, -2]$, and $\vec{\mathbf{b}}_2 = [4, 2, -3]$.

Answer

Let the direction vector be $\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2 = [1, 4, 5] \times [4, 2, -3] = [-22, 23, -14]$, then $\vec{\mathbf{x}}(t) = [1, 0, 2] + t[-22, 23, -14]$.

9. One point $\vec{\mathbf{p}}$ on the line and $\|$ to each of two given planes: $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, where $\vec{\mathbf{p}} = [1, 0, 2]$, $a_1 = 2$, $b_1 = -4$, $c_1 = 5$, $d_1 = 1$, $a_2 = 1$, $b_2 = -7$, $c_2 = 1$, and $d_2 = 4$.

Answer

Let the direction vector be $\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2 = [2, -4, 5] \times [1, -7, 1] = [31, 3, -10]$. Thus $\vec{\mathbf{x}}(t) = [1, 0, 2] + t[31, 3, -10]$.

10. Equidistant to two given planes (here we mean that each point on the line is equidistant to the two planes): $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, where

   a) $a_1 = 2$, $b_1 = -4$, $c_1 = 5$, $d_1 = 1$, $a_2 = 1$, $b_2 = -7$, $c_2 = 1$, and $d_2 = 4$.

   Answer

   The planes are not parallel because the normal vectors $\vec{\mathbf{b}}_1$ and $\vec{\mathbf{b}}_2$ are not multiples of each other. The point $\vec{\mathbf{p}} = [x, y, z]$ can be taken to be a point on both planes as shown in question #7, i.e. $\vec{\mathbf{p}} = [4, 1, -1]$. The direction vector $\vec{\mathbf{d}}$ can be taken to be the bisector of the 2 normal vectors:

$$\vec{\mathbf{d}} = \frac{\vec{\mathbf{b}}_1}{||\vec{\mathbf{b}}_1||} + \frac{\vec{\mathbf{b}}_2}{||\vec{\mathbf{b}}_2||} = \frac{[2, -4, 5]}{\sqrt{45}} + \frac{[1, -7, 1]}{\sqrt{50}}.$$

   Thus $\vec{\mathbf{x}}(t) = [4, 1, -1] + t\left(\frac{2}{\sqrt{45}} + \frac{1}{\sqrt{50}}, -\frac{4}{\sqrt{45}} - \frac{7}{\sqrt{50}}, \frac{5}{\sqrt{45}} + \frac{1}{\sqrt{50}}\right)$.

   b) $a_1 = -2$, $b_1 = 14$, $c_1 = -2$, $d_1 = 1$, $a_2 = 1$, $b_2 = -7$, $c_2 = 1$, and $d_2 = 4$.

   Answer

   First note that the normal vector $\vec{\mathbf{b}}_1$ is a multiple of $\vec{\mathbf{b}}_2$ ($\vec{\mathbf{b}}_1 = -2\vec{\mathbf{b}}_2$), hence the planes are $\|$. The point $\vec{\mathbf{p}}$ must be equidistant to the 2 planes so can be taken to be midway between any two points $\vec{\mathbf{p}}_1$ and $\vec{\mathbf{p}}_2$ each lying on one of the planes. Choose the point $\vec{\mathbf{p}}_1$ satisfying $-2x + 14y - 2z = 1$, say $\vec{\mathbf{p}}_1 = \left(-\frac{1}{2}, 0, 0\right)$ and choose $\vec{\mathbf{p}}_2$ satisfying $x - 7y + z = 4$, say $[4, 0, 0]$. The midpoint $\vec{\mathbf{p}}$ is:

$$\vec{\mathbf{p}} = \frac{1}{2}(\vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2) = \left(\frac{7}{4}, 0, 0\right).$$

The direction vector can be any nonzero vector $\vec{\mathbf{d}} = [d_1, d_2, d_3]$ such than $\vec{\mathbf{d}} \cdot \vec{\mathbf{b}}_1 = 0$, i.e. $-2d_1 + 14d_2 - 2d_3 = 0$. The choice $\vec{\mathbf{d}} = [1, 0, -1]$ works so $\vec{\mathbf{x}}(t) = \left(\frac{7}{4}, 0, 0\right) + t[1, 0, -1]$. Alternatively, $\vec{\mathbf{d}}$ could be taken to be $\vec{\mathbf{b}}_1 \times \vec{\mathbf{c}}$, where $\vec{\mathbf{c}}$ is any vector not $\|$ to $\vec{\mathbf{b}}_1$.

11. Equidistant to two given lines (here we mean that each point on the line is equidistant to the two given lines): $\vec{\mathbf{a}}_1 + t\vec{\mathbf{b}}_1$ and $\vec{\mathbf{a}}_2 + s\vec{\mathbf{b}}_2$.

    a) $\vec{\mathbf{a}}_1 = [2, 2, -2]$, $\vec{\mathbf{b}}_1 = [1, 4, 5]$, $\vec{\mathbf{a}}_2 = [2, 2, -2]$, and $\vec{\mathbf{b}}_2 = [4, 2, -3]$.
    b) $\vec{\mathbf{a}}_1 = [2, 2, -2]$, $\vec{\mathbf{b}}_1 = [1, 4, 5]$, $\vec{\mathbf{a}}_2 = [2, 4, -2]$, and $\vec{\mathbf{b}}_2 = [3, 12, 15]$.
    c) $\vec{\mathbf{a}}_1 = [2, 2, -2]$, $\vec{\mathbf{b}}_1 = [1, 4, 5]$, $\vec{\mathbf{a}}_2 = [2, 4, -2]$, and $\vec{\mathbf{b}}_2 = [4, 2, -3]$.

    Answer

    The direction vector $\vec{\mathbf{d}}$ in a), b), and c) can be found in the same way as the previous questions: the bisector between $\vec{\mathbf{b}}_1$ and $\vec{\mathbf{b}}_2$, i.e. $\vec{\mathbf{d}} = \frac{\vec{\mathbf{b}}_1}{||\vec{\mathbf{b}}_1||} + \frac{\vec{\mathbf{b}}_2}{||\vec{\mathbf{b}}_2||}$. It remains to find the point $\vec{\mathbf{p}}$ on the line which can be taken as the intersection point if the 2 given lines intersect.

    a) Since $\vec{\mathbf{a}}_1 = \vec{\mathbf{a}}_2$, $\vec{\mathbf{p}}$ is clearly [2,2,-2].
    b) Since the 2 lines are parallel we can take $\vec{\mathbf{p}}$ to be the midpoint between $\vec{\mathbf{a}}_1 = [2, 2, -2]$ and $\vec{\mathbf{a}}_2 = [2, 4, -2]$, i.e. $\vec{\mathbf{p}} = \frac{1}{2}(\vec{\mathbf{a}}_1 + \vec{\mathbf{a}}_2) = [2, 3, -2]$.
    c) We can take $\vec{\mathbf{p}}$ to be the intersection point of the 2 lines given by $[x_1, y_1, z_1] = [2 + t, 2 + 4t, -2 + 5t]$ and $[x_2, y_2, z_2] = [2 + 4s, 4 + 2s, -2 - 3s]$. Setting

    $$x_1 = x_2 \ \Rightarrow\ 2 + t = 2 + 4s \ \Rightarrow\ t = 4s.$$

    Now, set $y_1 = y_2$ and substitute $t = 4s$ to get

    $$2 + 16s = 4 + 2s \ \Rightarrow\ s = \frac{1}{7}.$$

    Substituting $s = \frac{1}{7}$ in $[x_2, y_2, z_2]$ gives:

    $$\vec{\mathbf{p}} = \left(\frac{18}{7}, \frac{30}{7}, -\frac{17}{7}\right).$$

Find an equation of a plane (if possible) given the following information:

1. One point $\vec{\mathbf{p}}$ on the plane and a normal vector $\vec{\mathbf{b}}$ to the plane, say $\vec{\mathbf{p}} = [1, 2, 3]$ and $\vec{\mathbf{b}} = [6, 5, 4]$.

   Answer

   $\vec{\mathbf{p}} = [1, 2, 3]$ and $\vec{\mathbf{b}} = [6, 5, 4]$, therefore the equation of the plane is $6(x-1)+5(y-2)+4(z-3) = 0$.

2. One point $\vec{\mathbf{p}}$ on the plane and $\perp$ to a line $\vec{\mathbf{a}} + t\vec{\mathbf{d}}$, say $\vec{\mathbf{p}} = [1, 0, 2]$, $\vec{\mathbf{a}} = [2, 4, -2]$, and $\vec{\mathbf{d}} = [4, 2, -3]$.

   Answer

   Let the normal vector $\vec{\mathbf{b}}$ to the plane be $\vec{\mathbf{b}} = \vec{\mathbf{d}} = [4, 2, -3]$, then the equation is $4(x-1) + 2(y-0) - 3(z-2) = 0$.

3. One point $\vec{\mathbf{p}}$ on the plane and $\|$ to another plane $ax + by + cz + d = 0$, say $\vec{\mathbf{p}} = [1, 0, 2]$, $a = 2$, $b = -4$, $c = 5$, and $d = 1$.

   Answer

   Let the normal vector be $\vec{\mathbf{b}} = [a, b, c] = [2, -4, 5]$, then the plane is given by $2(x-1) - 4(y-0) + 5(z-2) = 0$.

4. One point $\vec{\mathbf{p}}$ on the plane and $\perp$ to another plane $ax + by + cz + d = 0$, say $\vec{\mathbf{p}} = [1, 0, 2]$, $a = 2$, $b = -4$, $c = 5$, and $d = 1$.

   Answer

   The normal vector $\vec{\mathbf{b}}$ must be $\perp$ to the given normal vector $\vec{\mathbf{b}}_2 = [a, b, c] = [2, -4, 5]$. So let $\vec{\mathbf{b}}$ be any nonzero vector $\vec{\mathbf{b}} = [b_1, b_2, b_3]$ such than $\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}_2 = 0$, i.e. $2b_1 - 4b_2 + 5b_3 = 0$. The choice $\vec{\mathbf{b}} = [1, -2, -2]$ works so the plane is given by $(x-1) - 2(y-0) - 2(z-2) = 0$. Alternatively, $\vec{\mathbf{b}}$ could be taken to be $\vec{\mathbf{b}}_2 \times \vec{\mathbf{c}}$, where $\vec{\mathbf{c}}$ is any vector not $\|$ to $\vec{\mathbf{b}}_2$.

5. One point $\vec{\mathbf{p}}$ on the plane and $\perp$ to two given planes: $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, where

   a) $\vec{\mathbf{p}} = [1, 0, 2]$, $a_1 = 2$, $b_1 = -4$, $c_1 = 5$, $d_1 = 1$, $a_2 = 1$, $b_2 = -7$, $c_2 = 1$, and $d_2 = 4$.

      Answer

      The normal vector $\vec{\mathbf{b}}$ must be $\perp$ to both the given normal vectors $\vec{\mathbf{b}}_1 = [1, -4, 5]$ and $\vec{\mathbf{b}}_2 = [1, -7, 1]$. So let $\vec{\mathbf{b}}$ be $\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2 = [31, 4, -3]$, then the plane is given by $31(x-1) + 4(y-0) - 3(z-2) = 0$.

   b) $\vec{\mathbf{p}} = [1, 0, 2]$, $a_1 = -2$, $b_1 = 14$, $c_1 = -2$, $d_1 = 1$, $a_2 = 1$, $b_2 = -7$, $c_2 = 1$, and $d_2 = 4$.

      Answer

      Notice that the two given normal vectors $\vec{\mathbf{b}}_1 = [-2, 14, -2]$ and $\vec{\mathbf{b}}_2 = [1, -7, 1]$ are $\|$ so we can't use the method in a) because the cross product $\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2$ would give us zero. The normal vector $\vec{\mathbf{b}}$ must be $\perp$ to the normal vector $\vec{\mathbf{b}}_2 = [1, -7, 1]$. So let $\vec{\mathbf{b}}$ be any nonzero vector $\vec{\mathbf{b}} = [b_1, b_2, b_3]$ such than $\vec{\mathbf{b}} \cdot \vec{\mathbf{b}}_2 = 0$, i.e. $b_1 - 7b_2 + b_3 = 0$. The choice $\vec{\mathbf{b}} = [1, 0, -1]$ works so the plane is given by $(x-1) - (z-2) = 0$. Alternatively, $\vec{\mathbf{b}}$ could be taken to be $\vec{\mathbf{b}}_2 \times \vec{\mathbf{c}}$, where $\vec{\mathbf{c}}$ is any vector not $\|$ to $\vec{\mathbf{b}}_2$.

6. Three points $\vec{\mathbf{p}}_1$, $\vec{\mathbf{p}}_2$ and $\vec{\mathbf{p}}_3$ on the plane, say $\vec{\mathbf{p}}_1 = [2, -1, 3]$, $\vec{\mathbf{p}}_2 = [3, -1, 3]$ and $\vec{\mathbf{p}}_3 = [2, -1, 0]$.

   Answer

Let $\vec{\mathbf{p}}_1 = [2, -1, 3]$ be the point on the plane. Two vectors that lie in the plane ($\|$ to plane) are $\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1 = [1, 0, 0]$ and $\vec{\mathbf{p}}_3 - \vec{\mathbf{p}}_1 = [0, 0, -3]$, so take the normal vector $\vec{\mathbf{b}}$ to be:

$$\vec{\mathbf{b}} = (\vec{\mathbf{p}}_2 - \vec{\mathbf{p}}_1) \times (\vec{\mathbf{p}}_3 - \vec{\mathbf{p}}_1) = [0, 3, 0].$$

Therefore the equation of the plane is: $0(x-2)+3(y-(-1))+0(z-3) = 0 \Rightarrow 3(y+1) = 0$.

7. One point $\vec{\mathbf{p}}$ on the plane and a line $\vec{\mathbf{a}} + t\vec{\mathbf{d}}$ on the plane, say $\vec{\mathbf{p}} = [1, 0, 2]$, $\vec{\mathbf{a}} = [2, 4, -2]$, and $\vec{\mathbf{d}} = [4, 2, -3]$.

Answer

Two points on the plane are $[1, 0, 2]$ and $[2, 4, -2]$, thus $\vec{\mathbf{c}} = [1, 0, 2] - [2, 4, -2] = [-1, -4, 4]$ is a vector that lies on the plane. Since $\vec{\mathbf{d}}$ is another vector that lies on the plane, the normal vector $\vec{\mathbf{b}}$ must be $\perp$ to both $\vec{\mathbf{d}} = [4, 2, -3]$ and $\vec{\mathbf{c}} = [-1, -4, 4]$. So let $\vec{\mathbf{b}}$ be $\vec{\mathbf{d}} \times \vec{\mathbf{c}} = [-4, -13, -14]$, then the plane is given by $-4(x-1)-13(y-0)-14(z-2) = 0$.

8. Both of the lines: $\vec{\mathbf{a}}_1 + t\vec{\mathbf{b}}_1$ and $\vec{\mathbf{a}}_2 + t\vec{\mathbf{b}}_2$, are on the plane, where $\vec{\mathbf{a}}_1 = [2, 2, -2]$, $\vec{\mathbf{b}}_1 = [1, 4, 5]$, $\vec{\mathbf{a}}_2 = [3, 6, 3]$, and $\vec{\mathbf{b}}_2 = [4, 2, -3]$.

Answer

Two vectors lying in the plane are $\vec{\mathbf{b}}_1$ and $\vec{\mathbf{b}}_2$, thus the normal vector $\vec{\mathbf{b}}$ must be $\perp$ to both $\vec{\mathbf{b}}_1 = [1, 4, 5]$ and $\vec{\mathbf{b}}_2 = [4, 2, -3]$. So let $\vec{\mathbf{b}}$ be $\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2 = [-22, 23, -14]$. We can take either $\vec{\mathbf{a}}_1 = [2, 2, -2]$ or $\vec{\mathbf{a}}_2 = [3, 6, 3]$ to be a point on the plane, say $\vec{\mathbf{a}}_1 = [2, 2, -2]$. Therefore the plane is given by $-22(x - 2) + 23(y - 2) - 14(z + 2) = 0$.

9. One point $\vec{\mathbf{p}}$ on the line and $\perp$ to each of two given lines: $\vec{\mathbf{a}}_1 + t\vec{\mathbf{d}}_1$ and $\vec{\mathbf{a}}_2 + t\vec{\mathbf{d}}_2$, where $\vec{\mathbf{p}} = [1, 0, 2]$, $\vec{\mathbf{a}}_1 = [2, 2, -2]$, $\vec{\mathbf{d}}_1 = [1, 4, 5]$, $\vec{\mathbf{a}}_2 = [2, 4, -2]$, and $\vec{\mathbf{d}}_2 = [4, 2, -3]$.

Answer

The normal vector $\vec{\mathbf{b}}$ must $\|$ to both direction vectors $\vec{\mathbf{d}}_1$ and $\vec{\mathbf{d}}_2$. Since the two direction vectors are not $\|$ themselves, we cannot find a normal vector. Therefore, there is no plane that can be $\perp$ to both lines. If the lines were $\|$ then then it would be possible to find a plane $\perp$ to both lines (just take $\vec{\mathbf{b}}$ to be the direction vector of one of the lines.)

10. Equidistant to two given planes (here we mean that each point on the plane is equidistant to the two given planes): $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, where

a) $a_1 = 2$, $b_1 = -4$, $c_1 = 5$, $d_1 = 1$, $a_2 = 1$, $b_2 = -7$, $c_2 = 1$, and $d_2 = 4$.

Answer

The point $\vec{\mathbf{p}} = [x, y, z]$ can be taken to be a point on both planes, as was shown in question #7 in the previous section (equations of lines), where it was found that $\vec{\mathbf{p}} = [4, 1, -1]$. The normal vector $\vec{\mathbf{b}}$ can be taken to be the bisector of the 2 normal vectors:

$$\vec{\mathbf{b}} = \frac{\vec{\mathbf{b}}_1}{||\vec{\mathbf{b}}_1||} + \frac{\vec{\mathbf{b}}_2}{||\vec{\mathbf{b}}_2||} = \frac{[2, -4, 5]}{\sqrt{45}} + \frac{[1, -7, 1]}{\sqrt{50}}.$$

Thus the equation of the plane is:

$$\left(\frac{2}{\sqrt{45}} + \frac{1}{\sqrt{50}}\right)(x - 4) - \left(\frac{4}{\sqrt{45}} + \frac{7}{\sqrt{50}}\right)(y - 1) + \left(\frac{5}{\sqrt{45}} + \frac{1}{\sqrt{50}}\right)(z + 1) = 0.$$

b) $a_1 = -2$, $b_1 = 14$, $c_1 = -2$, $d_1 = 1$, $a_2 = 1$, $b_2 = -7$, $c_2 = 1$, and $d_2 = 4$.

Answer

First note that the normal vector $\vec{\mathbf{b}}_1$ is a multiple of $\vec{\mathbf{b}}_2$ ($\vec{\mathbf{b}}_1 = -2\vec{\mathbf{b}}_2$), hence the planes are $\|$. The point $\vec{\mathbf{p}}$ must be equidistant to the 2 planes so can be found in exactly the same way as in question #10 b) of the previous section, where it was found that

$$\vec{\mathbf{p}} = \left(\frac{7}{4}, 0, 0\right).$$

The normal vector can be taken to be either of the 2 given normal vectors, say $\vec{\mathbf{b}}_2 = [a_2, b_2, c_2] = [1, -7, 1]$, so the equation of the plane is:

$$\left(x - \frac{7}{4}\right) - 7(x-0) + (z-0) = 0.$$

11. Equidistant to two given lines: $\vec{\mathbf{a}}_1 + t\vec{\mathbf{d}}_1$ and $\vec{\mathbf{a}}_2 + t\vec{\mathbf{d}}_2$ (here we mean that each point on the plane is equidistant to the two lines), where

a) $\vec{\mathbf{a}}_1 = [2, 2, -2]$, $\vec{\mathbf{d}}_1 = [1, 4, 5]$, $\vec{\mathbf{a}}_2 = [2, 2, -2]$, and $\vec{\mathbf{d}}_2 = [4, 2, -3]$.

b) $\vec{\mathbf{a}}_1 = [2, 2, -2]$, $\vec{\mathbf{d}}_1 = [1, 4, 5]$, $\vec{\mathbf{a}}_2 = [2, 4, -2]$, and $\vec{\mathbf{d}}_2 = [3, 12, 15]$.

c) $\vec{\mathbf{a}}_1 = [2, 2, -2]$, $\vec{\mathbf{d}}_1 = [1, 4, 5]$, $\vec{\mathbf{a}}_2 = [2, 4, -2]$, and $\vec{\mathbf{d}}_2 = [4, 2, -3]$.

Answer

We can give the equation of a plane (there could be more than one!) if we have a point $\vec{\mathbf{p}}$ on the plane and 2 vectors $\vec{\mathbf{c}}_1$ and $\vec{\mathbf{c}}_2$ parallel to the plane (because then the normal vector is $\vec{\mathbf{b}} = \vec{\mathbf{c}}_1 \times \vec{\mathbf{c}}_2$.) The vector $\vec{\mathbf{c}}_1$ can be found in the same way in each part a), b), and c) as the bisector between $\vec{\mathbf{d}}_1$ and $\vec{\mathbf{d}}_2$, i.e. $\vec{\mathbf{c}}_1 = \frac{\vec{\mathbf{d}}_1}{||\vec{\mathbf{d}}_1||} + \frac{\vec{\mathbf{d}}_2}{||\vec{\mathbf{d}}_2||}$. The vector $\vec{\mathbf{c}}_2$ can also be found in the same way in a), b), and c) since the cross product $\vec{\mathbf{c}}_2 = \vec{\mathbf{d}}_1 \times \vec{\mathbf{d}}_2$. It remains to find the point $\vec{\mathbf{p}}$ on the plane which can be taken to be the intersection point between the 2 given lines if they intersect (as in #11 of the equation of lines exercises):

a) Since $\vec{\mathbf{a}}_1 = \vec{\mathbf{a}}_2$, $\vec{\mathbf{p}}$ is clearly [2,2,-2].

b) Since the 2 lines are parallel we can take $\vec{\mathbf{p}}$ to be the midpoint between $\vec{\mathbf{p}}_1 = [2, 2, -2]$ and $\vec{\mathbf{p}}_2 = [2, 4, -2]$, i.e. $\vec{\mathbf{p}} = \frac{1}{2}(\vec{\mathbf{a}}_1 + \vec{\mathbf{a}}_2) = [2, 3, -2]$.

c) We can take $\vec{\mathbf{p}}$ to be the intersection point of the 2 lines given by $[x_1, y_1, z_1] = [2+t, 2+4t, -2+5t]$ and $[x_2, y_2, z_2] = [2+4s, 4+2s, -2-3s]$. Setting

$$x_1 = x_2 \;\Rightarrow\; 2 + t = 2 + 4s \;\Rightarrow\; t = 4s.$$

Now, set $y_1 = y_2$ and substitute $t = 4s$ to get

$$2 + 16s = 4 + 2s \;\Rightarrow\; s = \frac{1}{7}.$$

Substituting $s = \frac{1}{7}$ in $[x_2, y_2, z_2]$ gives:

$$\vec{\mathbf{p}} = \left(\frac{18}{7}, \frac{30}{7}, -\frac{17}{7}\right).$$