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Fluctuation estimates for sub-quadratic gradient field actions

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In this article we estimate fluctuations of the scalar field ϕ for a special class of sub-quadratic actions which grow like $|\nabla\phi|^{2\alpha}$, $0 < \alpha < 1$. In particular if $\alpha = 1/2$ we show that in three dimensions $\langle e^{\gamma\phi_0}\rangle$ is bounded for γ small. For each edge (jk) we introduce an auxiliary field $t_{jk} \in \mathbb{R}$ to express the action as a superposition of Gaussian free fields. The effective action which arises from integrating over the Gaussian field is shown to be convex in t. The Brascamp-Lieb inequality is then applied to obtain the desired estimates on a nonuniformly elliptic Green's function.
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Dedicated to Elliott Lieb on the occasion of his 80th birthday

The theory of convex functions of the gradient of a scalar field is well developed and applies to problems arising from anharmonic crystals, dipole gases, and random surfaces. See for example Refs. 4 and 6. Let $\phi_j \in \mathbb{R}$ with $j \in \Lambda \subset Z^d$. In this article we consider a class of models described by finite volume measures which take the form

$$\text{(Normalisation)}\, e^{-\sum_j (V(\nabla\phi_j)+\epsilon\phi_j^2)} \prod_{j\in\Lambda} d\phi_j\,, \tag{1}$$

where $\epsilon > 0$ and for $\nabla\phi$ large,

$$V(\nabla\phi_j) = O\left(\left(\nabla\phi_j\right)^{2\alpha}\right)\,. \tag{2}$$

We will state results for the case of periodic boundary conditions on Λ and obtain estimates uniform in $\epsilon > 0$ as $\Lambda \to Z^d$.

There does not seem to be any mathematical literature which addresses models of this form with $0 < \alpha < 1$. We are especially interested in large deviations of the field, given by $\langle e^{\gamma\phi_0}\rangle$ when $\alpha = 1/2$ because a similar action appears in the measure describing linearly edge reinforced random walk (ERRW). See for example Ref. 9. For a class of models described below, we also bound moments of ϕ for any $0 < \alpha < 1/2$, in which case the action is not convex. We use the word *action* to denote the function in the exponent of a weight that multiplies Lebesgue measure, which in this case is $\sum_j V(\nabla\phi_j)$.

To define our model let Λ be a lattice torus Z_N^d where $N \geq 3$ is the side of the torus, which is an integer. In fact we make no use of symmetries. Let $\{j, k\}$ be an unordered pair of nearest neighbour sites in Λ. We denote such pair by $jk = \{j, k\}$ and let E be the set of all such pairs. For each $jk \in E$ we introduce an auxiliary field $t_{jk} \in \mathbb{R}$ and we define a Gaussian action,

$$A(\phi, t) = \sum_{jk\in E}\left(1 + \frac{1}{2}\beta(\phi_j - \phi_k)^2\right)e^{t_{jk}} + \frac{1}{2}\sum_{j\in\Lambda}\epsilon\phi_j^2\,, \qquad \beta, \epsilon > 0\,. \tag{3}$$

Then the partition function for our model is defined to be a superposition of Gaussians,

$$Z_\Lambda(\beta) = \int e^{-A(\phi,t)} \prod_{jk\in E} \left(e^{-e^{-\kappa t_{jk}} - \frac{1}{2}t_{jk}} dt_{jk}\right) \prod_{j\in\Lambda} d\phi_j\,, \quad \kappa > 0\,. \tag{4}$$

Let $\langle\cdot\rangle_\beta$ denote the expectation with respect to the probability measure on the space of $(\phi,\, t)$ configurations that is defined by the weight in the partition function (4). It depends on Λ and ϵ but since our estimates are uniform in Λ and ϵ we do not make these dependences explicit.

For $\kappa = 1$ the integral over $t = (t_{jk})$ can be explicitly evaluated (see (24) below) and the result, after dropping some factors of π, is

$$Z_\Lambda(\beta) = \int e^{-2\sum_{jk\in E}\sqrt{1+\frac{1}{2}\beta(\phi_j-\phi_k)^2} - \frac{1}{2}\sum_{j\in\Lambda}\epsilon\phi_j^2} \prod_{j\in\Lambda} d\phi_j\,, \tag{5}$$

which is a model as in (1) with $\alpha = \frac{1}{2}$. For $\kappa \neq 1$ we cannot evaluate the t integral explicitly, but by considering the t_{jk} that maximizes the exponent it is not hard to see that integration over t produces a model as in (1) such that $V(\nabla\phi_j)$ grows as $(\beta(\nabla\phi_j)^2)^{\kappa/(1\,+\,\kappa)}$ and hence $\alpha = \kappa/(1\,+\,\kappa)$.

Nonconvex potentials have also been studied by superposition in Ref. 5. In this case the potentials correspond to $\alpha = 1$, because the corresponding t variables have compact support. Furthermore, the superposition is uniformly elliptic whereas ours is not, and this enables us to get $\alpha < 1$.

Let $[\,v; w\,] = \sum_j v_j\, w_j$ denote the scalar product. Define the finite difference elliptic operator $D_\beta(t)$ by the quadratic form

$$[f; D_\beta(t) f] = \beta \sum_{jk\in E} (f_j - f_k)^2 e^{t_{jk}} + \epsilon \sum f_j^2 \tag{6}$$

and let $G(t; j, k)$ denote its Green's function $G(t) = (D_\beta(t))^{-1}$. Define $G_0 = (-\beta\Delta + \epsilon)^{-1} = D_0^{-1}$ by setting $t = 0$.

The ϕ field may be integrated out and, dropping factors of π, we obtain

$$Z_t = \int \det[D_\beta(t)]^{-1/2} \prod_{jk\in\Lambda} \left(e^{-e^{t_{jk}}}\, e^{-e^{-\kappa t_{jk}} - \frac{1}{2}t_{jk}} dt_{jk}\right). \tag{7}$$

Following terminology that is standard in physics we call this an *effective partition function*. Note

$$\langle\phi_0^2\rangle = \langle G(t; 0, 0)\rangle_t, \qquad \langle e^{\gamma\phi_0}\rangle = \langle e^{\gamma^2 G(t;0,0)/2}\rangle_t\,, \tag{8}$$

where the $\langle\cdot\rangle_t$ denotes the expectation over the auxiliary t fields. Defining $\phi\cdot v = \sum_j \phi_j v_j$ we have the slightly more general

$$\langle(\phi\cdot v)^{2n}\rangle = (2n-1)!!\,\langle[v; G(t)v]^n\rangle_t, \qquad \langle e^{\gamma\phi\cdot v}\rangle = \langle e^{\gamma^2[v;G(t)v]/2}\rangle_t\,. \tag{9}$$

Lemma 1: For all t and $\beta > 0$, $\ln\det[D_\beta(t)]$ is a convex function of t.

This lemma is a key element in our proofs. A similar determinant arises in the hyperbolic sigma model studied in Ref. 10 where partial results on convexity were proved. Lemma 1 follows directly from Kirchoff's theorem,[8] also known as the matrix tree theorem. The version we use can be found in Theorem 1 of Ref. 1. See also Theorem 1.19 of Ref. 7. The matrix tree theorem expresses $\det D(t)$ as a sum over weighted rooted forests with each nearest neighbor edge (j, k) assigned a weight $e^{t_{jk}}$ and each root a weight ϵ. Since $\det D(t)$ is a positive superposition of exponentials its logarithm is convex.

Lemma 1 implies that the effective action in the t variables is convex. In fact the Hessian H is bounded from below by the diagonal

$$\inf\left(e^t + \kappa^2 e^{-\kappa t}\right)\delta_{jk,lm} = c(\kappa)\delta_{jk,lm}\,, \tag{10}$$

where this defines the positive constant $c(\kappa)$.

Lemma 2: Let v denote a vector in R^Λ orthogonal to constants. The Green's function satisfies the quadratic form bound

$$0 \le [v; G(t)v] \le \sum_{jk\in E} ((G_0 v)_j - (G_0 v)_k)^2 e^{-t_{jk}} . \tag{11}$$

Proof: Without loss of generality we take $\beta = 1$. We can also set $\epsilon = 0$ because $[v; G(t)v]$ increases as $\epsilon \downarrow 0$. Let $D_0 = \nabla^*\nabla = G_0^{-1}$. According to (6) we can write $D_{\beta,t} = \nabla^* A^2 \nabla$ where A is the diagonal matrix whose entries on the diagonal are $e^{t_{jk}/2}$. By integration by parts followed by the Schwartz inequality followed by reversing the integration by parts

$$\begin{aligned}
[v; G(t)v] &= [v; G_0 D_0 G(t)v] = [\nabla G_0 v;\ \nabla G(t)v] \\
&= [A^{-1}\nabla G_0 v;\ A\nabla G(t)v] \\
&\le [A^{-1}\nabla G_0 v;\ A^{-1}\nabla G_0 v]^{1/2}[A\nabla G(t)v;\ A\nabla G(t)v]^{1/2} \\
&= [A^{-2}\nabla G_0 v;\ \nabla G_0(t)v]^{1/2}[v; G(t)v]^{1/2}
\end{aligned}$$

which, after dividing by $[v; G(t)v]^{1/2}$, is the desired inequality. □

We will now prove bounds on $\langle e^{-t_j}\rangle$ for any $\kappa > 0$ and for $\langle e^{\gamma\phi\cdot v}\rangle_\beta$ for suitable values of γ, β and $\kappa = 1$ assuming that $[v; G_0 v]$ is finite. In the proof of this second result we will prove that $\langle \exp(ae^{-t_j})\rangle < \infty$ for suitable values of a. As in Ref. 10 the method is a combination of a Ward identity and the Brascamp-Lieb bounds.[3]

Proposition 3: Let $\langle\cdot\rangle$ denote the expectation in t and ϕ defined via (4). For $\lambda, \kappa > 0$, there is a constant $C(\lambda, \kappa)$ such that

$$\langle e^{-\lambda t_{jk}}\rangle \le C(\lambda, \kappa) \tag{12}$$

and $\langle(\phi\cdot v)^{2p}\rangle$ is bounded by constants $\bar{C}(\kappa, p)$ provided $[v; G_0 v]$ is finite.

Proof: For λ positive or negative, let $q(\lambda) = \ln\langle e^{-\lambda t_{jk}}\rangle$. We bound $q(\lambda)$ using Taylor's theorem to second order in λ. To do this let $F_\lambda = e^{-\lambda t_{jk}}$ and let $\langle\cdot\rangle_\lambda = \langle\cdot F_\lambda\rangle/\langle F_\lambda\rangle$. Then

$$q(0) = 0 , \quad q'(0) = \langle -t_{jk}\rangle , \quad q''(\lambda) = \left\langle (t_{jk} - \langle t_{jk}\rangle_\lambda)^2\right\rangle_\lambda . \tag{13}$$

Since the action corresponding to $\langle\cdot\rangle_\lambda$ is also convex with Hessian bounded below as in (10), $q''(\lambda)$ is bounded by a constant $c(\kappa)$ using the Brascamp-Lieb bound [Ref. 3, Theorem 5.1]. By the bound and the Taylor expansion we have

$$\langle e^{-\lambda t_{jk}}\rangle \le e^{c(\kappa)^{-1}\lambda^2/2} e^{\lambda\langle -t_{jk}\rangle} . \tag{14}$$

To obtain a bound on $\langle -t_{jk}\rangle$ we use a Ward identity generated by the change of variables

$$t_{jk} \to t_{jk} + b. \tag{15}$$

Since the partition function does not depend on the constant b, the derivative with respect to b evaluated at $b = 0$ vanishes hence,

$$\left\langle -e^{t_{jk}}[1 + \frac{1}{2}(\phi_j - \phi_k)^2] + \kappa e^{-\kappa t_{jk}} - \frac{1}{2}\right\rangle = 0. \tag{16}$$

Referring to (6), by $[f; Gf] = \sup_\phi \left(2[f;\phi] - [\phi; D\phi]\right)$ and $[\phi; D\phi] \ge \beta e^{t_{jk}}(\phi_j - \phi_k)^2$ we have $\langle(\phi_j - \phi_k)^2\rangle = [(\delta_j - \delta_k); G(t)(\delta_j - \delta_k)] \le \frac{1}{\beta} e^{-t_{jk}}$, thus

$$\langle \kappa e^{-\kappa t_{jk}}\rangle \le \langle e^{t_{jk}}\rangle + \frac{1}{2} + \frac{1}{2\beta}. \tag{17}$$

Jensen's inequality implies $\kappa e^{-\kappa\langle t_{jk}\rangle} \le \left\langle e^{t_{jk}}\right\rangle + \frac{1}{2} + \frac{1}{2\beta}$ and by (14) with $\lambda = -1$,

$$\kappa e^{-\kappa\langle t_{jk}\rangle} \le e^{c(\kappa)^{-1}/2} e^{\langle t_{jk}\rangle} + \frac{1}{2} + \frac{1}{2\beta}. \tag{18}$$

Therefore, for some positive constant $C(\kappa)$,

$$\langle -t_{jk}\rangle \le C(\kappa)\,. \tag{19}$$

We insert this bound into (14) with $\lambda > 0$ and obtain (12).

To bound the moments we use (9), Lemma 2 and the Hölder inequality to bound the expectation of a product of $e^{-t_{jk}}$ factors by (12). □

Theorem 4: *Let $\langle\cdot\rangle_\beta$ be the expectation defined via (4) and let $\kappa = 1$ so that we are considering the model (5).* If v is orthogonal to constants and *if $\gamma^2[v; G_0 v] < 1$, then $\langle e^{\gamma\phi\cdot v}\rangle_\beta$ is uniformly bounded in ϵ and Λ.*

Proof: By (9) and Lemma 2 it suffices to estimate

$$\langle \exp(\frac{1}{2}\gamma^2 \sum_{jk\in E} a_{jk} e^{-t_{jk}})\rangle\,, \quad \text{with} \quad a_{jk} = [G_0(j,v) - G_0(k,v)]^2. \tag{20}$$

Note that $\sum_{jk\in E} a_{jk} = [v; G_0 v] \le 1$. For $\lambda \in [0, \frac{1}{2}\gamma^2]$ let $F_\lambda = \exp(\lambda \sum a_{jk} e^{-t_{jk}})$ and let $g(\lambda) = \ln\langle F_\lambda\rangle$. Expand g to second order in λ using the principle in (13) to express the derivatives with respect to λ in terms of a new expectation, $\langle\cdot\rangle_\lambda = \langle\cdot F_\lambda\rangle/\langle F_\lambda\rangle$. The effect of the additional factor F_λ reduces the convexity of the action expressed in (10) so that the Hessian corresponding to $\langle\cdot\rangle_\lambda$ is now bounded below by the diagonal quadratic form

$$(e^{t_{kj}} + e^{-t_{kj}} - \lambda a_{jk} e^{-t_{jk}})\delta_{jk,lm}\,. \tag{21}$$

These diagonal entries are bounded below by $(1-\lambda a_{jk})(e^{t_{jk}} + e^{-t_{jk}})$ and $\lambda a_{jk} \le \frac{1}{2}\gamma^2[v; G_0 v] \le 1/2$. From the Brascamp-Lieb bound [Ref. 3, Theorem 4.1], (21) and $\lambda a_{jk} \le \frac{1}{2}$,

$$\begin{aligned} 0 \le g''(\lambda) &= \langle \sum a_{jk} e^{-t_{jk}}; \sum a_{jk} e^{-t_{jk}}\rangle_\lambda \\ &\le \sum a_{jk}^2 \left\langle e^{-t_{jk}}\left[e^{t_{kj}} + e^{-t_{kj}} - \lambda a_{jk} e^{-t_{jk}}\right]^{-1} e^{-t_{jk}}\right\rangle_\lambda \\ &\le \sum a_{jk}^2 (1-\lambda a_{jk})^{-1} \langle e^{-t_{jk}}\rangle_\lambda \le 2\sum a_{jk}^2 \langle e^{-t_{jk}}\rangle_\lambda\,. \end{aligned} \tag{22}$$

Notice that the lower bound says that $g'(\lambda)$ is increasing. From this bound on $g''(\lambda)$ we get a bound on $g'(\lambda) = \sum_{jk\in E} a_{jk}\langle e^{-t_{jk}}\rangle_\lambda$ by integrating g'' with respect to λ:

$$\begin{aligned} 0 \le g'(\lambda) &= g'(0) + 2\int_0^\lambda \sum_{jk\in E} a_{jk}^2 \langle e^{-t_{jk}}\rangle_s ds \\ &\le g'(0) + 2[v; G_0 v]\int_0^\lambda \sum_{jk\in E} a_{jk}\langle e^{-t_{jk}}\rangle_s ds = g'(0) + 2[v; G_0 v]\int_0^\lambda g'(s) ds \\ &\le g'(0) + 2[v; G_0 v]\lambda g'(\lambda)\,. \end{aligned} \tag{23}$$

By the hypothesis and $\lambda < \frac{1}{2}\gamma^2$ we can solve this inequality to obtain $0 \le g'(\lambda) \le (1 - 2\lambda[v; G_0 v])^{-1} g'(0)$. Note that by Proposition 3, $0 \le g'(0) \le C(1,1)\,[v; G_0 v]$. Then we integrate this bound over $\lambda \in [0, \frac{1}{2}\gamma^2]$ and obtain the desired result. □

Proof of (5). We obtained the model (5) from the integral formula

$$e^{-2\sqrt{z}} = \frac{1}{\sqrt{\pi}} \int e^{-ze^t} e^{-e^{-t}-t/2} dt, \qquad z > 0, \tag{24}$$

with

$$z = 1 + \frac{1}{2}\beta(\phi_j - \phi_k)^2 . \tag{25}$$

To prove (24) we use the changes of variables $t \mapsto t - \ln\sqrt{z}$ and $u = e^t - e^{-t}$,

$$\begin{aligned}\int e^{-ze^t} e^{-e^{-t}-t/2} dt &= \int e^{-\sqrt{z}e^t} e^{-\sqrt{z}e^{-t}} e^{-t/2} \sqrt[4]{z} dt \\ &= \frac{1}{2}\int e^{-\sqrt{z}(e^t+e^{-t})} \left(e^{t/2} + e^{-t/2}\right) \sqrt[4]{z} dt, \\ &= \frac{1}{2}\int e^{-\sqrt{z}(e^{t/2}-e^{-t/2})^2 - 2\sqrt{z}} \left(e^{t/2} + e^{-t/2}\right) \sqrt[4]{z} dt \end{aligned} \tag{26}$$

$$= \int e^{-\sqrt{z}u^2 - 2\sqrt{z}} \sqrt[4]{z} du = \sqrt{\pi} e^{-2\sqrt{z}} \tag{27}$$

□

More general functions besides the square root can be represented as t integrals over Gaussians. A function $f : (0, \infty) \to \mathbb{R}$ is called completely monotonic if it is C^∞ and $(-1)^n f^{(n)}(x) \geq 0$, for $n \geq 0$, $x > 0$. A function $f : (0, \infty) \to [0, \infty)$ is called a Laplace exponent, if it is C^∞ and f' is completely monotonic. If f is a Laplace exponent then $e^{-tf(z)}$ can be written as an integral over $s \in [0, \infty)$ of e^{sz} with respect to a positive measure that depends on t. Laplace exponents include $f(z) = z^\alpha$, but for $\alpha \neq \frac{1}{2}$ we do not know if the weight in the integral representation is log concave as required by the Brascamp-Lieb bounds. See Section IX.11 of Ref. 11 and Ref. 2 for more background.

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