

COURSE 3

**A SHORT COURSE ON CLUSTER EXPANSIONS\***

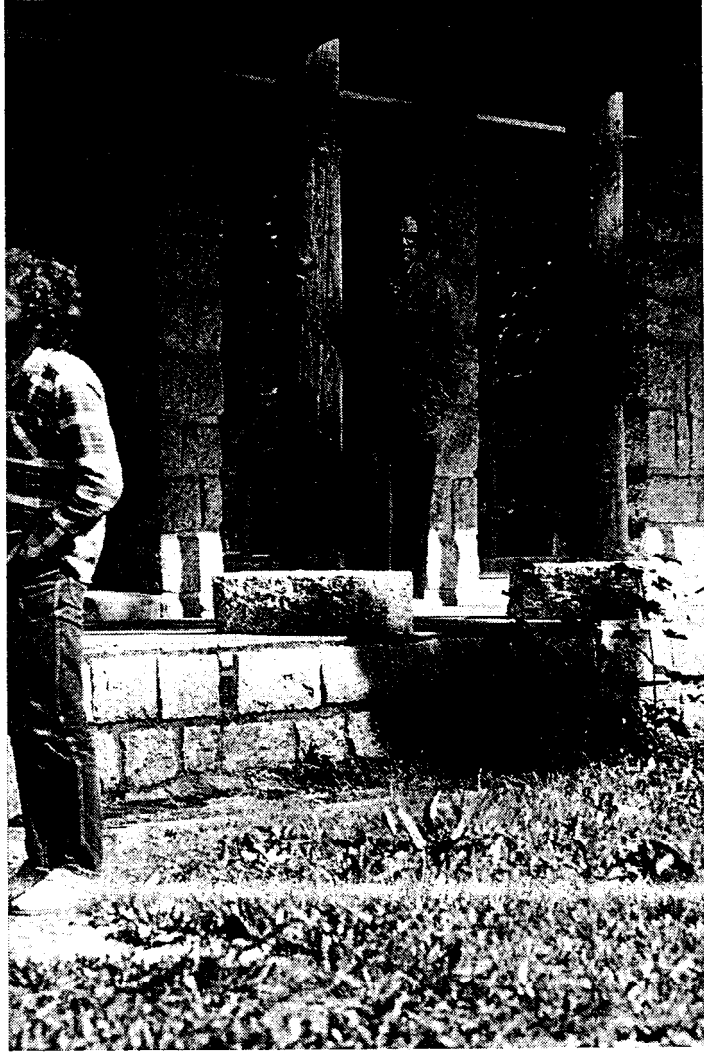
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## 1. Introduction

This is an introduction to the first four lectures of this course. These lectures are an attempt to provide some overview and review of the various kinds of expansions—Mayer expansions, high-temperature expansions, expansions about Gaussians—which are being used in mathematical work in statistical mechanics and quantum field theory. Another course will be devoted to lattice gauge field theory and will be given by John Imbrie.

To illustrate the theme of these lectures, consider a probability measure of the form

$$dP(\boldsymbol{\phi}) := Z^{-1} d\mu(\boldsymbol{\phi}) e^{-V(\boldsymbol{\phi})}, \quad (1.1)$$

where  $\boldsymbol{\phi} \equiv (\phi_i)$ ,  $i \in A$ , is a large collection of variables,  $d\mu$  is a probability measure with especially simple properties such as a product measure.  $V$  is a perturbation which ruins the factorisation properties of  $d\mu$ , for example

$$\langle \phi_i; \phi_j \rangle := \int dP \phi_i \phi_j - \int dP \phi_i \int dP \phi_j \neq 0.$$

We should like to formulate smallness conditions on  $V$  so that some remnant of factorisation should remain. For example the set  $A$  that labels the variables usually comes with some notion of distance, e.g., it is frequently a subset of a lattice  $\mathbb{Z}^d$  which is a discrete approximation to space-time and then if  $i, j \in A$  are well-separated lattice points, we hope to prove

$$\langle \phi_i; \phi_j \rangle \approx 0 \quad (\text{e.g. } \leq c_1 \exp(-c_2|i-j|)). \quad (1.2)$$

We should like to prove such statements with  $c_1, c_2$  independent of the size of  $A$ : indeed, in the sense of limits outside integrals,  $A$  is generally an infinite set,

The standard approach for investigating this type of question is to consider the logarithm of the partition function  $Z$

$$\log Z(\mathbf{h}) := \log \int d\mu \exp\left(-\mathbf{V} + \sum_{i \in \Lambda} h_i \phi_i\right),$$

where we have modified the original  $Z$  by adding the parameters  $\mathbf{h}$ . Taking the log tests the factorisation properties of  $Z$ : for example

$$\langle \phi_i; \phi_j \rangle = \frac{\partial^2}{\partial h_i \partial h_j} \log Z(\mathbf{h}) \Big|_{\mathbf{h}=\mathbf{0}}. \quad (1.3)$$

Notice that if  $\mathbf{V} = \mathbf{0}$ , multiple derivatives of  $\log Z$  with respect to distinct  $\mathbf{h}$ 's vanish because  $\log Z$  is a sum of functions of one variable. In this case we say  $\log Z$  is **local in  $\mathbf{h}$** .

The expansions described in these notes are of the form

$$\log Z(\mathbf{h}) = \sum_{X \subset \Lambda} K(X, \mathbf{h}), \quad (1.4)$$

where  $K(X, \mathbf{h})$  depends only on  $h_i$ ,  $\mathbf{i} \in X$ , is analytic in  $\mathbf{h}$  for all  $h_i$  small,  $\mathbf{i} \in X$ , and there is an estimate to the effect that, provided  $\mathbf{V}$  is small and approximately local, the sum converges for  $\mathbf{h}$  small. In particular, the terms  $K(X, \mathbf{h})$  are small when  $X$  is either large or very spread out so that factorisation properties such as (1.2) follow by differentiating with respect to  $h_i$  and  $h_j$ . In particular, an expansion of this form is a natural generalisation of the situation when  $\mathbf{V} = 0$  because then  $K(X, \mathbf{h})$  vanishes unless  $X$  is a single-element set, i.e.  $\log Z$  is local.

The expansion we have just discussed, where  $d\mu$  is a product measure, is generally called a **high-temperature expansion**. The underlying approximation,  $\mathbf{V} = 0$ , is a system of independent spins. For quantum field theory or the study of critical phenomena, the measure  $d\mu$  is not a product measure but a Gaussian measure so that we are trying to expand in corrections to a Gaussian. Conventional perturbation theory for quantum field theory accomplishes this, but generally is asymptotic rather than convergent. The expansions we describe are the closest we have so far been able to come to perturbation theory and still retain convergence. These expansions are often called **cluster expansions**.

The **Mayer expansion** is an expansion in corrections to the ideal gas. We suppose we have a bounded region  $\Omega \subset \mathbb{R}^d$ . An ideal gas in  $\Omega$  is a

measure on  $\mathcal{P}_0(\Omega) \equiv$  all finite sequences  $(x_1, \dots, x_N)$ ,  $N$  arbitrary, of  $\Omega$ . The measure is

$$Z_0^{-1} \frac{z^N}{N!} dx_1 \dots dx_N, \quad (\text{Counting measure in } \mathbb{N}).$$

The normalisation factor, which is known as the grand canonical partition function, is

$$Z_0 = \sum_{N=0}^{\infty} \frac{z^N}{N!} \int_{\Omega^N} dx_1 \dots dx_N,$$

where  $z$  is a parameter called the activity which determines the average  $N$  and thus the density  $N/\text{vol}(\Omega)$ . The Mayer expansion is an expansion for the log of a perturbed  $Z_0$ :

$$\log Z := \log \sum \frac{z^N}{N!} \int_{\Omega^N} d^N x e^{-V(x_1, \dots, x_N)}.$$

The Mayer expansion is a power series in  $z$ . It converges when  $|z|$  is small and  $V$  is small when evaluated on configurations  $x_1, \dots, x_N$  for which the differences  $x_i - x_j$  are all large.

An obvious generalisation of the ideal gas we have just presented is to replace  $(\Omega, dx)$  by an abstract measure space  $(\Omega, K)$  where  $K$  is a finite measure. A particularly important example, which we discuss in some detail, is called the polymer system:

$$Z := \sum \frac{1}{N!} \sum_{X_1, \dots, X_N \subset A} K(X_1) \dots K(X_N) e^{-V(X_1, \dots, X_N)},$$

where the ‘‘polymers’’  $X_i$  are summed over all finite subsets of a finite set  $A$  and  $K$  assigns a weight to each such set. Thus  $\Omega :=$  all finite subsets of  $A$ .  $V$  is frequently a **hard core interaction** which means  $V(X_1, \dots, X_N) = \infty$  if  $X_i \cap X_j \neq \emptyset$  for any  $i \neq j$ ,  $1 \leq i, j \leq N$ , otherwise  $v = 0$ .

The sects. 2-5 attempt to present all these expansions together with accurate estimates on convergence in a fairly unified manner using some recent combinatoric ideas due to Federbush and Battle. Sect. 6 is an illustration of the same philosophy, but there the expansions are being applied to analyse a self-avoiding random walk. This is actually a more complicated problem because the perturbation is not approximately

local and a naive high-temperature expansion does not converge. Thus sect. 6 is supposed also to illustrate the ideas of the renormalisation group [1] 'in a relatively simple context.

Unfortunately, gauge field theories present special problems which make the simple expansion about a Gaussian that we describe in sect. 4 not suitable. In sect. 5 we have made a not totally satisfactory attempt to explain these problems -and the best method presently available for dealing with them. This, section, then, is a prelude to John Imbrie's lectures.

Much of the material presented in these lectures has by now quite a long history and many applications. The most recent need for efficient methods of taking the logarithm will be seen in the lectures by Kupiainen and Gawedski, John Imbrie and others concerning the construction of effective actions in the renormalisation group. For background and further points of view on these expansions, we recommend the books by Seiler [2] and Glimm and Jaffe [3].

### 1.1. Notation and conventions

Our expansions will not make explicit mention of the parameters  $\mathbf{h}$  and  $z$  mentioned above. This is mostly because they can be assumed absorbed into the measures  $d\mu$  or their analogues in our arguments below.

We use the notation  $|X|$  to denote the cardinality of a set  $X$ . On a lattice  $\mathbb{Z}^d$  distances are also denoted by absolute values and in this case the norm equals the maximum component.

## 2. Connected parts, high-temperature expansions

Let  $X_w := \{1, 2, \dots\}$  label objects interacting with a Gibbs factor  $\psi := \exp(-V)$  defined on finite subsets. For example, frequently  $V$  has the **two-body form**:

$$V(X) = \frac{1}{2} \sum_{ij \in X} v_{ij}. \quad (2.1)$$

The starting point for high-temperature expansion is to define **connected parts**  $\psi_c \equiv (\exp(-V))$ , by:

$$\psi_c(X) := \psi(X), \quad \text{if } |X| = 1.$$

otherwise  $\psi$  is determined by recursively solving

$$\psi(X) := \sum_{\Pi \in \text{partitions of } X} \prod_{Y \in \Pi} \psi_c(Y). \quad (2.2)$$

The point of defining the connected parts is that they formally solve the problem of taking the log of a grand canonical partition function, e.g., if

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \int d\mu(x_1) \cdots d\mu(x_N) e^{-V(x_1, \dots, x_N)}, \quad (2.3a)$$

where  $d\mu$  is a finite measure on a space  $\Omega$  then (see appendix A)

$$\log Z = \sum_{N=0}^{\infty} \frac{1}{N!} \int d^N \mu(x_1, \dots, x_N) (e^{-V(x_1, \dots, x_N)})_c. \quad (2.3b)$$

At the moment nothing is being claimed about convergence. These statements only mean that if  $d\mu$  is replaced by  $z d\mu$  then the expansions hold as formal power series in  $z$ . This expansion is called the **Mayer expansion**.

For spin systems on a finite lattice  $\Omega$ , spins  $\phi = (\phi_x)_{x \in \Lambda}$ , the expansion of the Gibbs factor does not immediately yield the log but instead produces a polymer system: in this case, with  $\psi(\Lambda) = \exp(-V(\phi))$ ,

$$\begin{aligned} Z &= \int \prod_{x \in \Lambda} d\mu(\phi_x) \psi(\Lambda) = \sum_{\Pi \in \text{partitions of } \Lambda} \prod_{Y \in \Pi} \int d^Y \mu \psi_c(Y) \\ &= \sum_N \frac{1}{N!} \sum_{Y_1, \dots, Y_N, \cup Y_i = \Lambda} \prod_{i=1}^N \int d^{Y_i} \mu \psi_c(Y_i), \end{aligned}$$

where the  $Y$ 's are disjoint. If at the outset  $d\mu$  is normalised:

$$\int d\mu(\phi) e^{-V(Y)} = 1, \quad \text{if } |Y| = 1, \quad (2.4)$$

then some easy manipulations show that we can drop the condition  $\cup Y_i = \Lambda$  if at the same time we omit the  $Y$ 's with  $|Y| = 1$  from the sum, i.e.

$$\int d^{\Lambda} \mu \psi(\Lambda) = \sum_{\substack{Y_1, \dots, Y_N \subset \Lambda \\ \text{disjoint}}} \frac{1}{N!} \prod_{i=1}^N K(Y_i), \quad (2.5a)$$



where

$$K(Y) := \int d^Y \mu \psi_c(Y), \quad \text{if } |Y| > 1, \\ := 0, \quad \text{if } |Y| = 1. \quad (2.5b)$$

The right-hand side of (2.5a) is called a **polymer system**. In fact it is an example of a grand canonical ensemble as above in (2.3a) with  $\Omega := \{\text{finite subsets } \mathbf{C} \subset \Lambda\}$ . The disjointness of  $Y_1, \dots, Y_N$  is a hard core interaction:

$$\psi'(Y_1, \dots, Y_N) := e^{-V'(Y_1, \dots, Y_N)}, \\ V'(Y_1, \dots, Y_N) := \frac{1}{2} \sum_{i \neq j} v'(Y_i, Y_j), \quad (2.6)$$

and  $v'(X, Y) = 0$  if  $X \cap Y = \emptyset$ ,  $=\infty$  otherwise. Consequently we obtain

$$\log \int d^{\Lambda} \mu \psi(\Lambda) \\ = \sum_N \frac{1}{N!} \sum_{Y_1, \dots, Y_N \subset \Lambda} K(Y_1) \cdots K(Y_N) \psi'(Y_1, \dots, Y_N). \quad (2.7)$$

This is slightly confusing (because there are two  $\psi$ 's and  $\psi_c$ 's) example of “if at first you do not succeed, then expand and expand again”! This way of constructing the log of a spin-system partition function first appeared in [4].

These expansions are really systematic corrections to the ideal gas for (2.3) and the independent spin system  $\equiv$  product measure for (2.5). We will see that they are convergent when the interactions are small.

Neither expansion is directly relevant for perturbations “around a Gaussian”:

$$Z = \frac{1}{N} \int d^{\Lambda} \phi \exp\left(-\frac{1}{2} \sum_{xy} \phi_x A_{xy} \phi_y\right) \exp(-V(\phi)),$$

$N$  = normalisation for Gaussian part, because here we want an expansion in corrections to a Gaussian induced by the perturbation  $\mathbf{V}$ . (The expansions above would involve expanding the Gaussian part as well as  $V$ .)

### 3. Convergence of high-temperature expansions

The definition of connected part does not provide an effective method of calculation. The standard calculation procedure [5] is to verify that  $\psi_c(X)$  is the sum of connected graphs on  $X$ , e.g. for the case of two-body potentials (2.1), we prove in appendix A

$$(e^{-V(X)})_c = \sum_{\substack{G \in \text{connected} \\ \text{graphs on } X}} \prod_{ij \in G} (e^{-v_{ij}} - 1) \prod_{i \in X} e^{-\frac{1}{2}v_{ii}}. \quad (3.1)$$

This has the advantage of being closed form but it yields nonabsolutely convergent series for the log of the partition function because there are too many connected graphs in many situations. I will explain another representation. For each pair  $i, j \in X$  introduce a parameter  $s_{ij} \in [0, 1]$  and define a modified two-body potential by

$$V(X, s) := \frac{1}{2} \sum_{ij \in X} v_{ij} s_{ij}. \quad (s_{ij} = s_{ji}). \quad (3.2)$$

The introduction of such parameters can ruin desirable properties of  $V$  such as stability.

**Definition.**  $V$  is stable if  $V(X) \geq 0$  for all  $X$ .

The standard definition allows a lower bound by  $-\text{const. } |X|$ . Our definition is not less general because the diagonal parts of  $V$  can be adjusted to alter such a lower bound to zero (absorb the adjustments into a change of  $d\mu$  in (2.3a)).

**Definition.**  $V(X, s)$  is said to be a convex decoupling of  $V(X)$  if it is a convex linear combination of  $V(X)$  and potentials of the form  $\sum V(Y_i)$ , where  $\cup Y_i = X$  and the  $Y_i$ 's are disjoint.

If  $V$  is stable, any convex decoupling is stable. Self-energies (i.e., diagonal parts) are unchanged.

**Theorem 3.1.** *To each connected tree graph on  $X$  is associated a probability distribution  $dp$ , for  $s_{ij}$ ,  $i, j \in X$  so that for any  $V$ ,*

$$(e^{-V(X)})_c = \sum_{T \text{ on } X} \prod_{ij \in T} (-v_{ij}) \int_I dp_T(s) e^{-V(X, s)}$$

and  $dp$ , **is supported on  $s$  such that  $V(X, s)$  is a convex decoupling of  $V(X)$ .**

**Proof.** See appendix A.

Precursors to the representation appeared in [6] and [7]. Its present attractive and useful form is really due, to Battle and Federbush [8, 9]. The essential step in their arguments may be paraphrased as follows: suppose the representation of theorem 3.1 holds for some measures which are possibly not probability measures. Apply the identity to a potential  $V^{(T)}$  for which  $v_{ij} = 0$  unless  $ij \in T$  or  $i = j$ , where  $T$  is a fixed tree graph on  $X$ . Compare the resulting expression for  $(\exp -V^{(T)})_c$  with the one which results from using (3.1) to obtain

$$\prod_{i \in X} e^{-\frac{1}{2}v_{ii}} \prod_{ij \in T} (e^{-v_{ij}} - 1) = \prod_{ij \in T} (-v_{ij}) \int dp_T(s) e^{-V^{(T)}(X, s)}. \quad (3.3)$$

Now choose  $v_{ij} = \varepsilon$  if  $ij \in T$  or  $i = j$  and compare terms of leading order in  $\varepsilon$  as  $\varepsilon \searrow 0$  on both sides to obtain  $1 = \int dp_T(s) 1$ . This proves that the measures  $dp_T$  are probability measures.

**Corollary 3.2.**

(a) **If  $V$  is stable then**

$$|(e^{-V(X)})_c| \leq \sum_{T \text{ on } X} \prod_{ij \in T} |v_{ij}|.$$

(b) **If  $V$  is repulsive, i.e.  $v_{ij} \geq 0$  all  $i, j$  and  $v_{ii} = 0$ , all  $i$ ,**

$$|(e^{-V(X)})_c| \leq \sum_{T \text{ on } X} \prod_{ij \in T} |e^{-v_{ij}} - 1|.$$

**Proof.** The first part is an immediate consequence of the theorem. The second part is proven by replacing  $V(X, s)$  by  $V^{(T)}(X, s)$  in the right-hand side of the representation in the theorem. This makes the right-hand side bigger. Now apply (3.3). The second part was already known a long time ago [10, 11].

We now use these estimates to give simple proofs of convergence of Mayer expansions. For original references see ref. [12], p. 104. The arguments I present derive from ref. [7] and particularly from a paper by Cammarota [13]. See also ref. [2], p. 29.

**Convergence of the Mayer expansion.** We return to the grand canonical partition function (2.3a) and assume that  $V$  is stable so we can apply the bound (a) of the corollary so that

$$\begin{aligned} |\log Z| &= \left| \sum \frac{1}{N!} \int d^N \mu (e^{-V})_c \right| \\ &\leq \sum \frac{1}{N!} \sum_{T \text{ on } n \in \{1, \dots, N\}} \int d^N \mu \prod_{ij \in T} |v(x_i, x_j)| \\ &\leq \sum \frac{1}{N!} N^{N-2} \sup_T \int d^N \mu \prod_{ij \in T} |v(x_i, x_j)|, \end{aligned} \quad (3.4)$$

using Cayley's theorem (number of trees =  $N^{N-2}$ ). Now estimate the contribution of a tree graph,  $T$ , by stripping off its outermost branches, using

$$\int d\mu(y) |v(x, y)| \leq \sup_x \int d\mu(y) |v(x, y)| := q, \quad (3.5)$$

and working inwards to obtain

$$|\log Z| \leq Z \frac{N^{N-2}}{N!} \mu(\Omega) q^{N-1}, \quad (3.6)$$

$\mu(\Omega)$  is the volume-of-configuration space-factor that comes from the last integration done in rolling up the tree graph  $T$ . Evidently convergence is assured. if  $qe < 1$ , i.e.

$$e \sup_x \int d\mu(y) |v(x, y)| < 1. \quad (3.7)$$

Thus we have proved

**Theorem 3.3.** *The Mayer expansion (2.3b) converges if either*

- (a)  $V$  is **stable** and  $e \sup_x \int d\mu(y) |v(x, y)| < 1$ , **or**
- (b)  $V$  is **repulsive** and  $e \sup_x \int d\mu(y) |e^{-v(x, y)} - 1| < 1$ .

Case (b) looks as if it applies to the polymer Mayer expansion (2.7), but when the appropriate transcriptions are made one finds that (b) becomes

$$e \sup_{X \subset \Lambda} \sum_{\mathbf{Y}} |K(\mathbf{Y})| |e^{-v(X, \mathbf{Y})} - 1| < 1, \quad (3.8)$$

which is never satisfied for  $\Lambda$  large because of the supremum. Recall

$$\exp(-v(X, \mathbf{Y})) - 1 = 0, \quad \text{unless } X \cap Y \neq \emptyset.$$

Fortunately we can do the estimates more cautiously [13]. In the above proof of convergence we can do the sum over tree graphs in (3.4) by summing first over tree graphs with fixed incidence numbers  $\mathbf{d}_1, \dots, \mathbf{d}_N$ ,  $d_i :=$  number of lines meeting vertex  $\mathbf{i}$ , e.g.

$$\begin{aligned} \left| \sum_T (\cdot) \right| &= \left| \sum_{\mathbf{d}_1, \dots, \mathbf{d}_N} \sum_{T, \mathbf{d}} \sum_{\text{fixed}} (\cdot) \right| \\ &\leq \sum_{\mathbf{d}} \frac{(\mathbf{N} - 2)!}{(d_1 - 1)! \cdots (d_N - 1)!} \sup_{T, \mathbf{d} \text{ fixed}} |(\cdot)|. \end{aligned} \quad (3.9)$$

We have used Cayley's theorem on the number of tree graphs with incidence numbers  $\mathbf{d}_1, \dots, \mathbf{d}_N$ . The factors  $(d_i - 1)!$  in the denominator are an instance of the "extra factorials" found by Federbush and Battle [14]. With this extra step it is now easy to show that (3.8) improves to

$$Q := \sum_{d=0}^{\infty} \frac{1}{d!} \sup_{x \in \Lambda} \sum_{Y \ni x} |K(\mathbf{Y})| |Y|^d < 1. \quad (3.10a)$$

(See appendix A.) The factors  $|Y|$  count the ways polymers can overlap. In translation-invariant situations this is

$$\sup_x \sum_{Y \ni x} |K(\mathbf{Y})| e^{|Y|} < 1 \quad (3.10b)$$

and the **qualitative** form of this estimate cannot be improved because  $e^{|Y|}$  represents a partition function of small polymers swimming around inside a big polymer. A big polymer  $Y$  will not be stabilised against indefinite growth unless its activity  $\mathbf{K}(\mathbf{Y})$  is smaller than the partition function which represents what the system could put in its place, which is smaller polymers (with possibly negative activities, since we are proving a radius of convergence type theorem). It is of course possible that constants can be improved. See ref. [15] p. 110 for an example of better constants.

**Theorem 3.4 (implicit in [13]).** **If**

$$Z = \sum_N \frac{1}{N!} \sum_{\substack{X_1, \dots, X_N \subset \Lambda \\ \text{disjoint}}} K(X_1) \cdots K(X_N),$$

**then**

$$\begin{aligned} \log Z &= \sum_N \frac{1}{N!} \sum_{X_1, \dots, X_N \subset \Lambda} K(X_1) \cdots K(X_N) \psi_c(X_1, \dots, X_N) \\ &:= \sum_N I_N \end{aligned}$$

**satisfies the bound**

$$|I_N| \leq \frac{1}{N} Q^N |\Lambda|,$$

**where  $Q$  is defined in (3.10).**

It is possible by the same methods to prove more general (and useful!) theorems [16–18] where polymers  $X$  and  $Y$  have constituents that interact with each other. Here is one such theorem: Define a polymer  $X$  to be a collection of constituents,

$$X = \left( (x_1, x_2, \dots, x_p) \in (\Omega, \mu), \quad \mathbf{p} \text{ arbitrary} \right).$$

Let the interaction  $v'$  between two polymers be inherited from a **stable** interaction between its constituents,

$$v'(X, \mathbf{Y}) := \sum_{x \in X, y \in Y} v(x, y)$$

$$V'(X_1, \dots, X_N) = \frac{1}{2} \sum_{i, j=1, \dots, N} v'(X_{ij}, Y_j).$$

The Gibbs factor is

$$\psi'(X_1, \dots, X_N) := \exp(-V(X_1, \dots, X_N)).$$

Let  $d\mu'$  be a measure on the space of all polymers.

**Theorem 3.5.** *For the polymer system described above with*

$$Z = \sum_N \frac{1}{N!} \int d\mu'(X_1) \cdots d\mu'(X_N) \psi'(X_1, \dots, X_N),$$

*then*

$$\log Z = \sum_N \frac{1}{N!} \int d^N \mu' \psi'_c,$$

*with*

$$\frac{1}{N!} \int d^N \mu' |\psi'_c| \leq \frac{1}{N} \left( \mathfrak{q} \sup_{\mathbf{x}} \int d\mu(y) |v(x, y)| \right)^{N-1} \mu(\Omega),$$

*with*

$$\mathfrak{q} := \sum_d \frac{1}{d!} \sup_{\mathbf{x}} \left( \int_{X' \ni x} d\mu'(X) |X|^d \right).$$

**If  $V$  is repulsive then the  $v(x, y)$  may be replaced by  $|\exp(-v(x, y)) - 1|$  in the upper bound.**

This type of theorem was put to good use in refs. [16, 17] by G\"opfert and Mack. They realised that it is not necessary to expand all of a Gibbs factor into connected parts (by (2.2)) all at once. One can instead split  $V$  into short and long range parts,  $V = V_s + V_L$ , and expand only the Gibbs factor corresponding to  $V_s$ . This creates a new grand canonical partition function whose "particles" are the polymers created by the connected parts of  $\psi_c(V_s)$ . The constituents of these polymers are the original particles which continue to interact through the  $V_L$  Gibbs factor. In other words they get into the framework of theorem 3.5 which is applied to finally get the logarithm. This two-stage Mayer expansion (and more generally n-stage expansions) can give better regions of convergence than one-stage Mayer expansions, particularly on problems with several length scales.

Exercise. Prove theorem 3.5.

#### 4. Expansions about Gaussian measures

Consider a partition function of the form

$$Z(\Lambda) = \int d\mu(\phi) F(\Lambda),$$

where  $d\mu$  is a normalised Gaussian measure on variables  $\phi = (\phi_x)_{x \in \Lambda}$  mean zero and covariance  $C_{xy}$ .  $F(\Lambda)$  is a function of  $\phi$  with some factorisation properties: We assume  $\Lambda$  decomposes into blocks  $x' \in \Lambda'$  so that  $\Lambda = \cup_{x' \in \Lambda'} x'$  and  $F(\Lambda)$  has the form

$$F(\Lambda) \equiv F(\Lambda') = \prod_{x' \in \Lambda'} F_{x'}(\Phi_{x'}),$$

where  $\Phi_{x'} = (\phi_x)_{x \in x'}$ . All sets are finite. We assume that the functions  $F_{x'}$  have been normalised so that

$$\int d\mu(\phi) F_{x'}(\Phi_{x'}) = 1 \quad (\text{cf. (2.4)}). \quad (4.1)$$

**Cluster expansion.**

$$Z(\Lambda) \equiv \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\substack{Y_1, \dots, Y_N \subset \Lambda' \\ \text{disjoint}}} K(Y_1) \cdots K(Y_N), \quad (4.2)$$

$$K(Y') = \begin{cases} 0, & \text{if } |Y'| = 1, \\ \sum_T \int dp_T(s) \int d\mu_s \prod_{x'y' \in T} (\Delta_{x'y'}) F(Y'), & \text{otherwise,} \end{cases} \quad (4.3)$$

where:  $dp_T(s)$  is the probability measure of theorem 3.1 on  $s = (s_{x'y'})$ ,  $x'y' \in T$ ;  $d\mu_s$  is the Gaussian measure with covariance  $C_{xy}s_{xy}$  where  $s_{xy} = s_{x'y'}$  when  $x \in x'$ ,  $y \in y'$ ,  $s_{xy} = 1$  if  $x, y$  belong to the same block.

$$A_{x'y'} := \sum_{x \in x', y \in y'} \frac{\partial}{\partial \phi_x} C_{xy} \frac{\partial}{\partial \phi_y} \mathbf{a}. \quad (4.4)$$

**Proof.** We prove it for the special case where each  $F_{x'}$  is a polynomial. The identities then extend to a more general class of functions (such as  $F_{x'} \in C^\infty$  with exponential bounds on derivatives by continuity of each side of (4.2), (4.3) on such a class.

For  $F_{x'}$  a polynomial, all  $x'$ , we have

$$\int d\mu(\phi) F(\Lambda) = \exp\left(\frac{1}{2} \sum_{x'y' \in \Lambda} \Delta_{x'y'}\right) F(\Lambda') \Big|_{\phi=0}, \quad (4.5)$$

where the exponential is defined by its power series which automatically



truncates on  $F(\Lambda')$ . This relation is the familiar statement that the kernel of  $\exp(tA)$  is a Gaussian. We now set  $\exp(\frac{1}{2} \sum \Delta_{x'y'}) := \psi(\Lambda)$  and expand it into connected parts by (2.2). (All operators commute, all functions of  $\Delta_{x'y'}$  are defined by power series which truncate on polynomials.)

By repeating the analysis (2.4)–(2.7) for spin-systems we obtain (4.2) with  $K(Y')$  given by

$$\begin{aligned} K(Y') &:= \left( \exp \left( \frac{1}{2} \sum_{x'y' \in Y'} \Delta_{x'y'} \right) \right)_c F(Y') \Big|_{\phi=0} \\ &\equiv \psi_c(Y') F(Y') \Big|_{\phi=0}. \end{aligned}$$

We now express  $\psi_c(Y')$  using theorem 3.1 in terms of trees and  $\exp(\frac{1}{2} \sum s_{x'y'} \Delta_{x'y'})$  which we eliminate in favor of  $d\mu_s$  by using (4.5) with  $d\mu \rightarrow d\mu_s$ . The result is the formula for  $K(Y')$ , (4.3).

This cluster expansion is easy to use: as a pedological illustration we specialise to the case where the blocks  $x'$  are single sites in  $\mathbf{A}$  so we may as well write  $F_{x'}(\Phi_{x'}) = F_x(\phi_x)$ . We make the following assumptions: (4.1) and

$$\text{I: } \left| \frac{\partial^r}{\partial \psi^r} F_x(\phi) \right| \leq \lambda^r r!, \quad \text{all } x, \quad r = 1, 2, \dots, \quad \text{all } \phi \text{ real,}$$

$$\text{II: } \sup_x \sum_y |C_{xy}| \leq c,$$

then for  $\Lambda$  sufficiently small we can construct  $\log \mathbf{Z}(\Lambda)$  as an expansion convergent no matter how large  $\Lambda$  is. We do this by showing that the polymer gas given by the cluster expansion of  $\mathbf{Z}(\Lambda)$  has a convergent Mayer expansion (cf. (2.7)) by virtue of theorem 3.4. This last theorem requires us to estimate

$$Q \equiv \sum_{d=0}^{\infty} \frac{1}{d!} \sup_z \sum_{Y \ni z} |K(Y)| |Y|^d,$$

with  $\mathbf{K}$  as given by the cluster expansion (4.3) and show that for  $\Lambda$  small  $Q < 1$ .

By our assumption on  $F_x$  and (4.3)

$$|K(Y)| \leq \sum_{T \text{ on } Y} \prod_{xy \in T} |C_{xy}| \prod_{x \in Y} (\lambda^{d_x} d_x!),$$

where  $d_x$  is the incidence number of  $T$  at  $x$ . We interchange the sum over  $Y$  and  $T$  in  $Q$ . This means we sum over all trees on  $\{1, 2, \dots, N\}$  and precede this sum by a sum over  $Y = \{z, x_2, \dots, x_N\}$ . We use assumption II to sum over  $Y$ . The result is

$$Q \leq \sum_{N=2}^{\infty} \frac{1}{(N-1)!} \sum_{\text{Tree } T \text{ on } \{1, \dots, N\}} \prod_{i=1}^N (\lambda^{d_i} d_i!) C^{N-1} e^N.$$

The encouraging factorial is there because the **set**  $Y$  gets counted  $(N-1)!$  times by summing over its possible elements. Now we estimate the sum over  $T$  using Cayley's theorem (cf. (3.9)) and find

$$Q \leq \sum_{N=2}^{\infty} \frac{1}{N-1} C^{N-1} e^N \sum_{d_1, \dots, d_N} \prod_{i=1}^N (\lambda^{d_i} d_i!).$$

From this it is clear that  $Q < 1$  if  $A$  is small depending on  $C$  so  $\log Z(X)$  has a convergent expansion for  $A$  small.

It is also possible to obtain convergent cluster expansions when the functions  $F_i$  are not analytic, indeed in many applications cluster expansions are preceded by  $C^\infty$  partitions of unity in the space of  $\phi$ . If, for example, we assume that there is a  $p > 1$  so that

$$\left| \frac{\partial^r}{\partial \phi^r} F_x(\phi) \right| \leq \lambda^r (r!)^p \quad (4.6)$$

then the discussion above fails. In this case more information about the covariance  $C_{xy}$  is apparently necessary. The typical assumptions would be

- (i)  $A$  is a finite subset of a lattice  $\mathbb{Z}^d$ ,
- (ii)  $|C_{xy}| < C \exp(-m|x-y|)$ .

With these assumptions tree graphs with a vertex  $x$  with a very large incidence number  $d_x$  are small despite the  $(d_x!)^p$  factor arising from derivatives of  $F_i$  because, **for any  $p$**

$$\sum_{y_1, \dots, y_{d_x} \in \mathbb{Z}^d} \prod_{k=1}^{d_x} |C_{xy_k}| \leq C_p^{d_x} (d_x!)^{-p} \quad (4.7)$$

where  $y_1, y_2, \dots, y_{d_x}$  are summed over all **distinct**  $d_x$ -tuples of sites in  $\mathbb{Z}^d$ . This inequality (which is easy to prove) says that if the covariance has

better than integrable decay then a vertex with large incidence number forces a small contribution because it is connected to  $d_x$  **distinct** sites and many of them have to be far away. Versions of this principle remain valid even if the covariance does not decay exponentially. Factorials of the type in (4.6) are also produced when cluster expansions are “improved” by combining with some renormalisations procedure such as integration by parts.

**Exercise.** Let  $A$  be a set partitioned into blocks  $x' \in A'$ . Prove the conditions

$$|\partial_\phi^r F_{x'}(\Phi_{x'})| \leq 4 \lambda^{|r|} r!,$$

where  $r = (\mathbf{r})$ ,  $x \in x'$ , is any nonzero multi-index,  $\phi$  is real

$$\sup_{x'} \sum_{\mathbf{y}'} \sup_{x \in x', y \in y'} |C_{xy}| \leq C_2,$$

$$e C_2 \lambda (1 - \lambda)^{-|x'|} \leq \mathbf{1} \text{ for all } x' \in A',$$

$$\int d\mu(\phi) F_{x'}(\Phi) = 1$$

are sufficient to guarantee the convergence of the expansion for  $\log Z(A)$  obtained by the Mayer expansion of the polymer system produced by this “block-wise” cluster expansion.

**Remarks.** The above problem is only meant to be illustrative. For example as we have seen, we can permit  $(|r|!)^p$ , if a stronger assumption on  $C_{xy}$  is imposed. Also we can replace the bound by  $\lambda^{|r|} r!$  which is uniform in  $\phi$  by a function which grows exponentially in  $\phi$  or even  $\phi^2$  (with a small coefficient).

For the benefit of those readers with some expertise in constructive field theory we close this section with an outline of the application of this cluster expansion to the two-dimensional  $:P(\phi):$  Euclidean quantum field theory. This was in fact the problem which originally inspired this expansion [6, 7].

**Cluster expansion for a continuum model.**  $A$  is a rectangle in  $\mathbb{R}^2$ , divided into disjoint unit squares called blocks and denoted by  $x' \in A'$ . The partition function is

$$Z(\Lambda) := \int d\mu(\phi) \exp\left(-\int_\Lambda dx :P(\phi(x)):\right),$$

where  $P$  is a semi-bounded polynomial and  $d\mu$  is the Euclidean free field

whose covariance is  $C(x - y) := (1 - \Delta)^{-1}(x - y)$ . We apply (4.2), (4.3) and need to estimate  $Q$ , just as before. By (4.1) the constant in  $P$  must be chosen so that  $Z(\text{block}) = 1$ . We find

$$|K(Y)| \leq \sum_T \sup_s \left| d\mu_s \exp\left(-\int_Y :P:\right) \prod_{x', y' \in T} \int_{x'} dx \int_{y'} dy \right. \\ \left. \times C(x - y) \left(\frac{\partial}{\partial \phi(x)} - \frac{\partial}{\partial \phi(x)} :P:\right) \left(\frac{\partial}{\partial \phi(y)} - \frac{\partial}{\partial \phi(y)} :P:\right) \right|,$$

$d\mu_s$  has a covariance of the form  $C(x, y, s) = C(x - y)s_{xy}$  where  $s_{xy} = 1$  if  $x, y$  belong to the same block and otherwise equals  $s_{x'y'} \in [0, 1]$ . The properties of  $dp_T(s)$  guarantee that  $C(s)$  is positive-definite and thus is the covariance of a Gaussian measure, which in fact defines  $d\mu_s$ . We now apply the Cauchy-Schwarz inequality and obtain

$$|K(Y)| \leq \sum_T \sup_s \left( \int d\mu_s \exp\left(2 \int_Y :P:\right) \right)^{1/2} \\ \times \left\{ \int d\mu_s \prod_{x', y' \in T} \left( \int_{x'} dx \int_{y'} dy C(x - y) \left(\frac{\partial}{\partial \phi(x)} - \frac{\partial P}{\partial \phi(x)}\right) \right. \right. \\ \left. \left. \times \left(\frac{\partial}{\partial \phi(y)} - \frac{\partial P}{\partial \phi(y)}\right) \right)^2 \right\}^{1/2} \\ \leq \sum_T \prod_{x', y' \in T} e^{-\text{dist}(x', y')} \prod_{x' \in Y'} (d_x!)^p e^{c|Y'|}, \quad (4.8)$$

where we have obtained the  $\exp(c|Y'|)$  using stability and conditioning, see ref. [3] pp. 157, 196, on the term with the exponential. The conditioning argument is applicable because the convex decoupling property of  $dp_T(s)$  “lifts” Neumann-Dirichlet bracketing of  $(1 - \Delta)^{-1}$  to its  $s$  dependent analogues. Furthermore, if  $\mathbf{A}$  has b.c.’s on the boundaries of blocks then  $C(x, y, s) = C(x, y, 1) = (1 - \Delta)^{-1}$ . Returning to (4.8), the  $(d_x!)^p$  factors come from calculating the second  $d\mu_s$  integral by Wick’s theorem and bounding the graphs by  $L_p$  norms of  $C(x, y, s)$  as explained on page 156 of ref. [3]. The  $\exp(-\text{dist}(x', y'))$  factors come from the exponential decay of  $C(x, y)$  which immediately implies decay for  $C(x, y, s)$ . From this bound to a proof that  $Q < 1$  if the coefficients of  $P$  are small (so that  $c$  is large negative) enough is an exercise with Cayley’s theorem and (4.7) of the same type we have already discussed.

**Exercise.** (Certain field theories have a convergent perturbation theory for the pressure, e.g. sine-Gordon theories. Here we will give an approach to a simple theorem of this type.)

Consider

$$Z = \int d\mu(\phi) e^{-\lambda V}, \quad V = \sum_{x \in \Lambda} v(\phi_x),$$

where we assume that the covariance of the Gaussian measure  $d$  satisfies

$$\sup_x \sum_y |C_{xy}| < \infty$$

and the perturbation  $v$  is bounded and analytic in a strip centred on the real axis. The problem is to prove that  $|\Lambda|^{-1} \log Z$  has a convergent Taylor series in  $\lambda$  about  $\lambda = 0$ , uniformly in  $\Lambda$ . We use the following method:

$$Z = \sum \frac{(-\lambda)^N}{N!} \int d\mu V^N = \sum \frac{(-\lambda)^N}{N!} e^{\frac{1}{2}\Delta} V^N \Big|_{\phi=0}.$$

The  $\partial/\partial\phi$  derivatives inside  $\Lambda$  can act on any of the  $N$  factors of  $V$ , so following a trick introduced by Feynman (in: Lectures in Physics) we ‘mechanise’ Leibniz’ rule by writing

$$\frac{\partial}{\partial\phi_x} = \sum_{i=1}^N \frac{\partial}{\partial\phi_x^{(i)}}$$

for each  $\partial/\partial\phi_x$  in  $\Lambda$ .  $\partial/\partial\phi_x^{(i)}$  acts only on the  $i$ th factor of  $V$ . After this substitution,  $\mathbf{A} = \sum \Delta_{ij}$  is a sum of ‘two-body’ operators with each of the  $N$  factors of  $V$  being ‘bodies’. We obtain an expansion for  $\log Z$  in the usual way by passing to the connected part of  $\exp(\frac{1}{2} \sum \mathbf{A}_i)$ , as above, and this is written in terms of tree graphs and estimated using the hypotheses on  $v$  along the lines of our examples above.

**Remark.** Since the coefficients of the series for  $\log Z$  in the exercise are, by definition, **truncated expectations**, the exercise provides a bound on truncated expectations of the following type

$$|\langle V^{(1)}, \dots, V^{(N)} \rangle| \leq \sum_T \sup_s \int d\mu_s \prod_{ij \in T} (\Delta_{ij}) V^{(1)} \dots V^{(N)}$$

(where  $V^{(1)}, \dots, V^{(N)}$  can be different functions).

## 5. Ericc cluster expansions

There are situations, such as in gauge theories, where the partition function cannot be written globally as a single perturbed Gaussian measure. This can lead to expansions in which different terms are approximated by different Gaussians. In this case the cluster expansion

of the last section is not so convenient because all polymers in it are still coupled through their common dependence on the covariance of the Gaussian which will be summed over.

We will now describe another type of expansion which is better suited to this situation. This type of expansion was first invented by Glimm, Jaffe and Spencer [19]. The one we will explain is a modification of their scheme which is convenient for lattice theories but probably not good for continuum regularisations. In the context of the **block spin** renormalisation group [1] it is all right since each stage is a unit-lattice problem.

After we have described these types of expansions and conditions for convergence we give an example which is supposed to illustrate in a reasonably simple way the situation in gauge theories where it is impossible to describe the possible field configurations at a point or a bond with one coordinate patch in field space, unless there are coordinate singularities. The example is the matter field part of the lattice Abelian Higgs model in the unitary gauge at weak coupling, i.e., nearly Gaussian. Despite the term “weak coupling”, this analysis is far from sufficient to obtain results valid in the continuum limit. This will be the subject of later lectures in this series by John Imbrie. An analysis of the complete unit-lattice Abelian Higgs model at weak coupling is given in ref. [20].

We consider a normalised Gaussian measure  $d\mu$  on  $(\phi_x)$ ,  $x \in \mathbf{A}$ , with covariance  $C_{xy}$ , which is given by

$$d\mu(\phi) := Z_0^{-1} d^{\mathbf{A}}\phi e^{-S_0(\mathbf{A})},$$

$$S_0(\mathbf{A}) := \frac{1}{2} \sum_{xy \in \mathbf{A}} A_{xy} (\phi_x - \phi_y)^2 + \frac{1}{2} m^2 \sum_{x \in \mathbf{A}} \phi_x^2 - \sum_{x \in \mathbf{A}} h_x \phi_x, \quad A_{xy} \geq 0.$$

With the notation of sect. 4 we perturb  $d\mu$  by a function  $F$  which factorises across blocks:

$$F(\mathbf{A}) \equiv F(\Lambda') = \prod_{x' \in \Lambda'} F_{x'}(\Phi_{x'}), \quad (5.1)$$

where

$$\mathbf{A} = \bigcup_{x' \in \Lambda'} \mathbf{x}'.$$

We introduce  $d\mu_s$  by replacing  $\mathbf{A}_{,,}$  by  $A_{xy}(s)$  in numerator **and denominator** of  $d\mu$  with

$$A_{xy}(s) := s_{x'y'} A_{xy}, \quad \text{if } x \in x', y \in y', x' \neq y',$$

$$:= A_{xy}, \quad \text{if } xy \in \text{same block}$$

When  $s = (s_{x'y'}) = 0$ , all blocks are decoupled.

The partition function we are interested in is

$$Z(\Lambda) := \int d\mu(\phi) F(\Lambda'),$$

$Z(\Lambda, s)$  is defined by replacing  $d\mu$  by  $d\mu_s$ .

**Cluster expansion**

$$Z(\Lambda) = Z(\Lambda, \mathbf{0}) \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\Gamma_1, \dots, \Gamma_N, \text{ disjoint}} K(\Gamma_1) \cdots K(\Gamma_N), \quad (5.2)$$

**if  $\Gamma$  is disconnected  $K(\Gamma) = 0$ , otherwise**

$$K(\Gamma) := \int ds, \partial^\Gamma Z(\Lambda, s_\Gamma) / Z(\Lambda, \mathbf{0}), \quad (5.3a)$$

**$\Gamma$  is a graph made up of bonds  $x'y'$ , i.e.,**

$$\Gamma \subset \{x'y' : x'y' \in \mathbf{A}\},$$

$$s_\Gamma := (s_{x'y'}) \text{ with } s_{x'y'} = \mathbf{0} \text{ if } x'y' \notin \Gamma$$

$$\partial^\Gamma := \prod_{x'y' \in \Gamma} \frac{\partial}{\partial s_{x'y'}}.$$

**Proof.** By scaling each  $F(x')$  by a constant we can assume without loss of generality that for each  $x'$ ,  $Z(x') = 1$  which implies  $Z(\Lambda, \mathbf{0}) = 1$ . By the fundamental theorem of calculus

$$Z(\Lambda) = 1 + \sum_{\Gamma} \int ds, \partial^\Gamma Z(\Lambda, s_\Gamma)$$

Now we observe that if  $\Gamma$  decomposes into connected components  $\Gamma_1, \dots, \Gamma_r$ , then  $\partial^\Gamma Z$  factorises and, because we have arranged  $Z(x') = 1$ ,

the factorisation may be written in the form

$$\prod_1^N \partial^{f_i} Z(\Lambda, s_{\Gamma_i}),$$

(5.2) and (5.3a) follow immediately. End of proof.

Define  $X(\Gamma)$  to be the set of blocks linked by  $\Gamma$ , i.e.,

$$X(\Gamma) := \bigcup \{z' : z' = x' \text{ or } y', \text{ some } x'y' \in \Gamma\}$$

then we can also write the formula for  $K(\Gamma)$ , with  $\Gamma$  connected, as

$$K(\Gamma) := \int ds_{\Gamma} \partial^{\Gamma} Z(X(\Gamma), s_{\Gamma}) / Z(X(\Gamma), \mathbf{0}), \quad (5.3b)$$

which displays its locality property. This is true because all  $s_{x'y'}$  with  $x'y' \notin \Gamma$  are zero so that numerator and denominator of (5.3a) factor across  $X(\Gamma)$  and the factors external to  $X(\Gamma)$  cancel.

It is more standard to assume that the underlying Gaussian measure,  $d\mu$  has mean zero, i.e.  $\mathbf{h} = 0$ . This can always be achieved by translating  $\phi$ , e.g.  $\phi_x \mapsto \phi_x + \int d\mu \phi_x$ , before doing a cluster expansion. This leads to more accurate bounds than what we are doing when  $\mathbf{h}$  is large. Our reason for not doing a translation is that the expansion performs more ‘localisation’: if you translate first,  $K(\Gamma)$  will depend on  $\int d\mu \phi$  where  $d\mu$  is the original Gaussian measure instead of the localised one with  $s$  parameters.

This type of expansion does more localisation than the expansion of sect. 4 because all covariances are localised by Neumann-type boundary conditions on  $\partial X(\Gamma)$ . The price to be paid is a more complicated convergence proof.

Another point to notice is that  $d\mu$  and  $d\mu_s$  need not have been Gaussian measures: any measure which factorises suitably when  $s$  factors are zero could pass through the proof that (5.2) and (5.3) hold. The special feature of Gaussian measures is that there is an effective method (integration by parts) to evaluate the multiple  $s$  derivatives.

For our example it will be necessary to consider also expansions for Gaussian normalisation factors, which we now present. This expansion will also illustrate some of the ideas needed to bound the cluster expansion.



**Expansion of Gaussian normalisation factors.** In the definition of  $d\mu(\phi)$  and  $d\mu_s(\phi)$  there is a normalisation factor

$$Z_0(\Lambda) = \int d^{\Lambda} \phi e^{-S_0(\Lambda)} .$$

We will expand this into “local” quantities:

$$\begin{aligned} Z_0(\Lambda) &= Z_0(\Lambda, s = 1) = Z_0(\Lambda, \mathbf{0}) \exp(\log Z_0(\Lambda, \mathbf{1}) - \log Z_0(\Lambda, \mathbf{0})) \\ &= Z_0(\Lambda, \mathbf{0}) \exp\left(\sum_{\Gamma} W(\Gamma)\right), \end{aligned} \quad (5.4)$$

with.

$$W(\Gamma) := \int ds_{\Gamma} \partial_s^{\Gamma} \log Z_0(\Lambda, s\mathbf{r}) . \quad (5.5)$$

A priori  $\Gamma$  must be summed over all graphs but since  $Z_0$  factors if  $\Gamma$  is not connected,  $W(\Gamma) = 0$  if  $\Gamma$  is not connected.

It is possible to evaluate  $W(\Gamma)$ . We perform the  $s$  derivatives, which, by definition of truncation, leads to a truncated expectation: if  $\Gamma = \{b_1, b_2, \dots, b_N\}$  with  $b_i = x'_i y'_i$ ,

$$W(\Gamma) = \int ds_{\Gamma} \left\langle \frac{\partial}{\partial s_{b_1}} S_0; \frac{\partial}{\partial s_{b_2}} S_0; \dots; \frac{\partial}{\partial s_{b_N}} S_0 \right\rangle_{s_{\Gamma}}$$

where  $\langle \rangle_{s_{\Gamma}}$  denotes the expectation with respect to  $d\mu_{s_{\Gamma}}$ . Since  $\langle \rangle$  is Gaussian this expectation can be evaluated as a sum of connected Feynman graphs. The graphs have either the topology of a single closed loop or a line whose two external legs end on magnetic fields. For the standard situation where

$$\mathbf{A} \subset \mathbb{Z}^d, \mathbf{A}' := \{\text{blocks: } L \times \dots \times L \text{ boxes } \subset \mathbf{A}\}$$

$$A_{xy} = 1, \quad \text{if } xy \text{ are nearest neighbors,}$$

$$= 0, \quad \text{otherwise,}$$

$$m^2 > 0,$$

we can depict a typical  $\Gamma$  and a Feynman graph  $G$ , a loop, contributing to  $W(\Gamma)$  as in fig. 1 (for  $d = 2$ ).

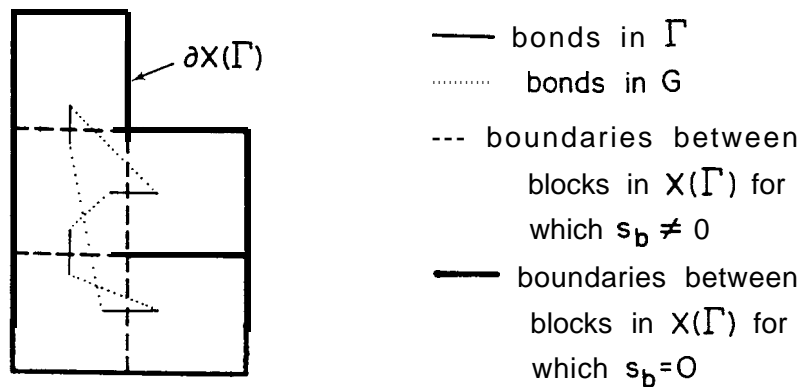


Fig. 1.

From fig. 1 it is clear that summing overall Feynman diagrams, which are loops which contribute to  $W(T)$ , is the same as summing over all walks which visit each bond in  $\Gamma$  and return to a specified starting bond with a weight  $C_{xy}$ ,  $x \in ax'$ ,  $x'$  in one bond,  $y \in dy'$ ,  $y'$  in the other, assigned to each step. This consideration makes it easy to see that the sum over all graphs (including open lines ending on external fields) is small when  $mL \gg 1$ , because the covariance  $C_{xy}$  decays according to  $\exp(-(m-\varepsilon)|x-Y|)$  uniformly in  $s$ , [20] sect. 12. Thus if we define an upper bound  $A_4$  on the magnetisation near  $\Gamma$  by

$$M := \sup\{|\langle \phi_x \rangle_s| : s, x \text{ next to a face crossed by } \Gamma\},$$

then, [20] sect. 13, for  $L$  large enough, so that  $mL \gg 1$ ,

$$\begin{aligned} |W(\Gamma)| &\leq (1 + M^2) \exp(-c_d mL(|\Gamma|)), & \text{if } |\Gamma| > d, \\ &\leq c_m L^{d-1} (1 + M^2), & \text{if } |\Gamma| \leq d. \end{aligned} \quad (5.6)$$

If blocks of different sizes are used, the same bounds hold with  $L$  replaced by the minimum  $L$  in  $\Gamma$ .

**Convergence for Erice expansion (5.2) and (5.3).** We assume

$$\begin{aligned} m > 0, \quad \Lambda \subset \mathbb{Z}^d, \\ A_{xy} &= 1, \quad \text{if } xy \text{ are nearest neighbors,} \\ &= 0, \quad \text{otherwise.} \end{aligned} \quad (5.7)$$

Such assumptions are natural for lattice field theories.  $F$  is small in the sense that for some  $\mathbf{p}$ ,  $c_1$ ,  $c_2$  and  $\lambda$  (small),

$$\sup_{\Phi} |\partial_{\Phi}^r F_{x'}| \leq \lambda^{|\mathbf{r}|} (|\mathbf{r}|!)^p \prod_{x \in x'} c_1 \exp|c_2 \phi_x|, \quad (5.8)$$

where  $\mathbf{r} \neq 0$  is a multi-index:  $\mathbf{r} = (r_x)_{x \in x'}$ ,  $|\mathbf{r}| = \sum_{x \in x'} r_x$ .

The magnetic field  $\mathbf{h}$  is bounded according to

$$\sup_x |h(x)| e^{-m \text{dist}(x, \text{faces crossed by } \Gamma)} \leq 1. \quad (5.9)$$

This guarantees a bounded magnetisation near  $\Gamma$ . We assume the block size  $L$  is large so

$$mL \gg 1. \quad (5.10)$$

With such assumptions the conclusion is

$$\left| \int ds_{\Gamma} \partial^{\Gamma} Z(X(\Gamma), s_{\Gamma}) \right| \leq A e^{-c_3 |\Gamma| + c_4 |X(\Gamma)|}, \quad (5.11)$$

where  $c_3$  can be made arbitrarily large positive by first taking the blocks large and then taking  $A$  small. To convert this to a bound on the polymer activity  $K(\Gamma)$  we need a bound on  $Z^{-1}(X(\Gamma), 0)$ . If the blocks are uniformly bounded in size this can usually be obtained by noting that this is a product of  $Z^{-1}(x')$  factors and these ordinarily are easy to bound uniformly in  $\lambda$  as  $\lambda \rightarrow 0$ .

**Principles of proof of (5.11).** We refer the reader to ref. [20] sect. 14, for a detailed proof of (5.11). The left-hand side of (5.11) is less than

$$\sup_{s_{\Gamma}} \left| \int d\mu_{s_{\Gamma}} F(X(\Gamma)) \prod_{x', y' \in \Gamma} (;I_{x', y'}) \right|, \quad (5.12)$$

where  $X(\Gamma)$  is the union of blocks linked by  $\Gamma$ , i.e.,

$$X(\Gamma) := \bigcup \{z' : z' \in \{x', y'\} \in \Gamma\},$$

and  $I_{x', y'}$  is the term in the exponential in  $d\mu_s$  which gets “pulled down” when  $\partial/\partial s_{x', y'}$  is performed,

$$I_{x'y'} \equiv \frac{1}{2} \sum_{\substack{x \in x' \\ y \in y'}} A_{xy} (\phi_x - \phi_y)^2 \equiv \frac{\partial}{\partial s_{x'y'}} S_0 .$$

Now recall that the normalisation  $N$  in  $d\mu$  is also made  $s$  dependent and **also gets differentiated**. This has the important effect of **truncating the expectation** of a product of  $I$ 's. This is recorded in the notation by the semi-colon in (5.12). For the special case that  $F$  is a polynomial, the truncation would mean that when Wick's theorem is used to evaluate the right-hand side of (5.12), only graphs such that every connected subgraph contains an  $F$  vertex can contribute. The rest get cancelled. The graphs are on " $I$ " vertices, each having two legs, and  $F$  vertices, i.e.,  $\phi$  derivatives of  $F$ . Such graphs will be small if:

- (1) derivatives of  $F$  are small.
- (2) large  $h_x$ 's occur only far from  $\Gamma$ . These graphs can have external legs ending on factors  $h_x$ . We have to be sure such factors do not overwhelm the small factors from derivatives of  $F$ .
- (3) The blocks must be large so that the factors  $I_{x'y'}$  are far apart on the scale  $m^{-1}$ . Then when the  $I$  factors are linked up in a connected graph, its vertices will be joined up by covariances whose exponential decay in distance between vertices will provide small factors to compensate for the number of graphs (and  $h$ 's). The same mechanism is needed in (2).

This last feature is different from the expansion of sect. 4. In this expansion the covariance is being localised at the same time as the perturbation and so it is not surprising that we need a condition that guarantees this is only being done on a length scale which is large in comparison with the  $m^{-1}$ , which is the correlation length of  $d\mu$ .

When  $F$  is not a polynomial, the truncated correlation can still be calculated by integration by parts in the form

$$\langle G\phi_x \rangle = \langle G \rangle \langle \phi_x \rangle + \sum_y \langle \phi_x; \phi_y \rangle \langle \partial G / \partial \phi_y \rangle ,$$

which holds for  $\langle \cdot \rangle$  the expectation for any Gaussian measure

$$\langle \phi_x; \phi_y \rangle \equiv \langle \phi_x \phi_y \rangle - \langle \phi_x \rangle \langle \phi_y \rangle .$$

This integration by parts can be used to reduce all the fields in the  $I$ 's in the truncated expectation. End of discussion.

Before passing on to an example of these expansions at work we need to illustrate one other frequently occurring idea which is used to unite several different expansions with mutual compatibility conditions into one polymer system.

Consider a polymer system where the polymer is labelled, for example, by a graph with nearest-neighbor bonds on blocks and a variable  $s$  (for **species**) so that

$$Z = \sum_N \frac{1}{N!} \sum_{(\Gamma_1, s_1), \dots, (\Gamma_N, s_N)} \prod_{i=1}^N K(\Gamma_i, s_i).$$

To each configuration  $(\Gamma, s)$  we associate the smallest union of blocks which contains all blocks linked by some  $\Gamma \in \Gamma$ . We decompose this union into components  $C_1, \dots, C_M$  where each  $C$  is a union of blocks which is connected in the sense that the blocks in  $C_i$ ,  $i$  fixed, touch along faces,  $\dots$ , edges, corners, but no two different  $C_i$ 's touch each other. We sum over  $\Gamma, s$  by first summing over configurations of  $\Gamma, s$  compatible with  $C_1, \dots, C_M$  and then summing over  $C$  configurations, i.e.,

$$Z = \sum_M \frac{1}{M!} \sum_{C_1, \dots, C_M \subset \Lambda} \bar{K}(C_1) \cdots \bar{K}(C_M),$$

where no two  $C$ 's may touch and

$$\bar{K}(C) := \sum_N \frac{1}{N!} \sum_{\substack{\Gamma_1, s_1, \dots, \Gamma_N, s_N \\ \text{compatible with } C}} \prod_{i=1}^N K(\Gamma_i, s_i).$$

Frequently there are also compatibility conditions between the  $(\Gamma, s)$ , e.g. hard cores, restrictions on which  $s$  species can touch which, etc.  $\dots$  If there are conditions of this type, they go into the sum which defines  $\bar{K}(C)$ . The nice feature of  $\bar{K}(C)$  is that, although the original compatibility conditions could have been complicated, they are all replaced by the simple condition that no two  $C$ 's may touch.

It is also possible to allow the  $\Gamma$ 's a finite range interaction of range strictly less than  $L$  because since  $C$ 's are unions of blocks in  $\mathbf{A}'$  the condition that they not touch separates them by a distance  $\geq L$  and they cannot interact by the short range force.

It is necessary to be able to prove bounds of the form

$$|\bar{K}(C)| \leq e^{-\bar{c}|C'|},$$

with  $\bar{c}$  large enough to permit taking the log by a Mayer expansion. This can be done as follows. Define, for  $\alpha \geq 0$ ,

$$K_\alpha(\Gamma, s) := e^{\alpha|\Gamma|} |K(\Gamma, s)|$$

and choose  $\alpha$  so large that

$$\inf \left\{ \sum_1^N |\Gamma_i| : N \text{ arbitrary, } \Gamma_1, \dots, \Gamma_N \text{ compatible with } C \right\} \alpha \\ \geq (\bar{c} + 1)|C'|.$$

This is possible because a configuration  $\Gamma_1, \dots, \Gamma_N$  is compatible with  $C$  only if the union of the blocks linked by all the  $\Gamma$ 's covers (in fact equals)  $C$ ,

$$\bigcup_1^N X(\Gamma_i) = C,$$

so if  $C$  is large, so is  $\sum |\Gamma_i|$ . Then it follows from the definition of  $\bar{K}$  that

$$|\bar{K}(C)| \leq e^{-(\bar{c}+1)|C'|} Z(K \rightarrow K_\alpha).$$

We now need conditions on  $K$  so that  $Z(K \rightarrow K_\alpha)$  is less than  $\exp(|C'|)$  and then we have the desired bound on  $\bar{K}(C)$ . A standard procedure is to write

$$Z(K \rightarrow K_\alpha) \equiv \sum \frac{1}{N!} \sum_{\Gamma, s} \prod_1^N K_\alpha(\Gamma_i, s_i) \leq \exp \left( \sum_{\Gamma, s: rcc} K_\alpha(\Gamma, s) \right) \\ \leq \exp \left( |C'| \sup_{x' \in C'} \sum_{\Gamma, s: \Gamma \text{ links } x'} K_\alpha(\Gamma, s) \right),$$

which shows that

$$\bar{K}(C) \leq e^{-\bar{c}|C'|}$$

is implied by  $Z(K_\alpha) \leq \exp|C'|$ , which in turn is implied by

$$\sup_{x'} \sum_{\substack{\Gamma, s \\ \Gamma \text{ links } x'}} |K(\Gamma, s)| e^{\alpha|\Gamma|} \leq 1, \quad (5.13)$$

for  $\alpha$  large depending on  $\bar{c}$ .

Before leaving this we also mention that convergence of sums such as the one in (5.13) when  $\Gamma$  is a graph of nearest-neighbor bonds on blocks forming a regular partition of  $\Lambda$ , is implied by

$$|K(\Gamma)| \leq e^{-c|\Gamma|}$$

with  $c$  sufficiently large, because the number of  $\Gamma$  with  $|\Gamma|$  fixed, linking a given block, is less than  $\exp(\text{const.}|\Gamma|)$ , [19] p. 219. Moreover this implies that the number of sets  $X'$  which are connected unions of blocks containing a given  $x'$ , with  $|X'|$  fixed, is also less than  $\exp(\text{const.} |X'|)$ , [19] p. 219. This means that sums of the form

$$\sum_{X' \ni x'} |K(X')| \exp|X'|,$$

which come up in checking the convergence criterion for the Mayer expansion applied to polymer systems, likewise converge if  $|K(X')| \leq \exp(-c|X'|)$  with  $c$  large enough.

**Example.** (See also [20] .) We will discuss the application of these ideas to the following model:

$$Z := \int_{\Omega} d^A \phi e^{-S(\Lambda)},$$

where

$$\Omega := [-\rho_0, \infty)^A \text{ with } \rho_0 = O(\lambda^{-1/2}), \Lambda \subset \mathbb{Z}^d,$$

$$S(\Lambda) := \frac{1}{2} \sum_{xy \in \Lambda} (\phi_x - \phi_y)^2 + \frac{1}{2} m^2 \sum_{x \in \Lambda} \phi_x^2 + \lambda \sum_{x \in \Lambda} \phi_x^4.$$

We have chosen this peculiar looking model because it contains a standard feature of gauge theory: a field-space singularity (at  $\phi = \mathbf{p}$ ). In Imbrie's lectures we will see that for the Higgs model in the unitary gauge, the matter field is of this type.

The object is to study this model when  $\Lambda$  is small. We will see that with

the aid of the cluster expansion, the expansion for Gaussian normalisations and the principle of grouping several expansions into C's, we can transform this system into a polymer system whose polymer activities are exponentially bounded in a manner that permit us to take the logarithm using the Mayer expansion as in sect. 3. As we have stressed in the introduction, such a representation immediately implies properties such as exponential decay of truncated correlations.

We decompose  $A$  into blocks  $x' \in \mathbf{A}'$  which are  $L \times \dots \times L$  boxes with  $L$  an odd integer. We define a field  $\phi$  to be small at site  $x$  if

$$|\phi_x| \leq p(\lambda) \equiv |\log \lambda|^{d+1}.$$

Regions where this is satisfied are candidates for a Gaussian approximation. We isolate the bad regions by a partition of unity of the form

$$1 = \sum_{\Lambda_0} \chi_{\Lambda_0} \zeta_{\Lambda_0^c},$$

where  $\chi$  is a smoothed characteristic function of the event that  $\phi$  is small inside  $\mathbf{A}$ ,  $\zeta$  is a smoothed characteristic function of the event:  $\phi$  is such that  $\Lambda_0^c$  is the smallest union of blocks which contains all large field sites and the fields are small on the sites next to  $\partial\Lambda_0^c$ . We will denote a generic connected component of  $\Lambda_0^c$  by  $R_0$  and refer to  $R_0$  as a **hole**. We back off from these holes by introducing a set  $\Lambda_1^c \supset \Lambda_0^c$  which is the smallest union of blocks such that

$$(a) \text{dist}(\partial\Lambda_0, \partial\Lambda_1) \geq r(\lambda) := \log^2 A,$$

$$(b) \text{the connected components of } \Lambda_1^c \text{ are separated by a distance } \geq L.$$

A connected component  $R_1$  of  $\Lambda_1^c$  may of course contain more than one hole; we shall refer to  $R_1$  as a **super-hole**.

The following bound will be used to show that superholes become arbitrarily rare as  $\lambda \rightarrow 0$

$$\sum_{RCR} Z_0^{-1}(R_1) \int d^{R_1} \phi e^{-S_0(R_1)} \zeta_R \leq 4^{|R_1|} e^{-cr(\lambda)^{-d} p^2(\lambda) |R_1|}, \quad (5.14)$$

$R$  is summed over all sets which are unions of holes inside  $R_1$  **and compatible with  $R$** .

This bound is obtained by scaling all fields in the numerator by  $\phi \rightarrow 2\phi$  so that  $S_0 \rightarrow 4S_0 = S_0 + 3S_0$ . The factor  $\exp(-3S_0)$  is very small, for all  $\mathbf{R} \subset R_1$  because of  $\zeta_R$ . The  $\mathbf{R}$  sum has less than  $2^{|R_1|}$  terms. (5.14)



follows. The bound shows that for any fixed  $L$  the right-hand side goes to zero as  $A \rightarrow 0$  faster than any power of  $A$ .

After inserting the partition of unity we can write  $Z$  in the form

$$Z = \sum_{\Lambda_0 \subset \Lambda} \int_{\Omega} d^{\Lambda_0^c} \phi e^{-S(\Lambda_0^c)} \zeta_{\Lambda_0^c} \int d^{\Lambda_0} \phi e^{-S^*(\Lambda_0)} \chi_{\Lambda_0}.$$

The asterisk means that couplings across  $\partial \Lambda_0$  are included.

In preparation for cluster expansions away from the holes, we define a normalised Gaussian measure by

$$d\mu_{\Lambda_0}^*(\phi) := (Z_0^*(\Lambda_0))^{-1} d^{\Lambda_0} \phi e^{-S_0^*(\Lambda_0)}.$$

This measure does not have mean zero because it feels the fields inside the holes: the couplings across  $\partial \Lambda_0$  act like a magnetic field at the boundary  $\partial \Lambda_0$ . (The measure is conditioned at  $\partial \Lambda_0$ .) Thus

$$Z = \sum_{\Lambda_0 \subset \Lambda} \int_{\Omega} d^{\Lambda_0^c} \phi e^{-S(\Lambda_0^c)} \zeta_{\Lambda_0^c} Z_0^*(\Lambda_0) \Xi^*(\Lambda_0),$$

with

$$\Xi^*(\Lambda_0) := \int d\mu_{\Lambda_0}^* \exp\left(-\lambda \sum_{x \in \Lambda_0} \phi_x^4\right) \chi_{\Lambda_0}.$$

Our expansions are applied to  $\Xi^*(\Lambda_0)$  and  $Z_0^*(\Lambda_0)$  at **fixed values** of  $\phi$  inside  $\Lambda_0$ . Thus the objects being expanded live in  $\Lambda_0$ , rather than  $\Lambda$ ; the expansions are based on the following types of block: each superhole  $R_1$  is a block, the rest of  $\Lambda_0$ , namely  $\Lambda_0 \setminus R_1$ , is filled with  $L \times \dots \times L$  blocks, see fig. 2. This choice ensures that no blocks close to  $\Lambda_0$  are being decoupled and hence that the magnetisation caused by the conditioning at  $\partial \Lambda_0$  is  $O(\lambda^N)$  for any  $N$  (of order  $\lambda^{-mr(\lambda)}$ ) near any contour generated by the expansion.

The expansion for  $Z_0^*$  is

$$Z_0^*(\Lambda_0) = Z_0 \frac{Z_0^*(\Lambda_1^c \setminus \Lambda_0^c)}{Z_0(\Lambda_1)} e^{\Sigma(W^*(\Gamma) - W(\Gamma))}$$

This is a formula that follows easily by inserting expansions for  $Z_0^*(\Lambda_0)$  and  $Z_0 \equiv Z(\Lambda)$ . Notice that  $(W^* - W)(\Gamma) = 0$  unless  $\Gamma$  links  $\Lambda_1^c$ . Another point is that the  $Z^*/Z$  ratio factors, corresponding to the

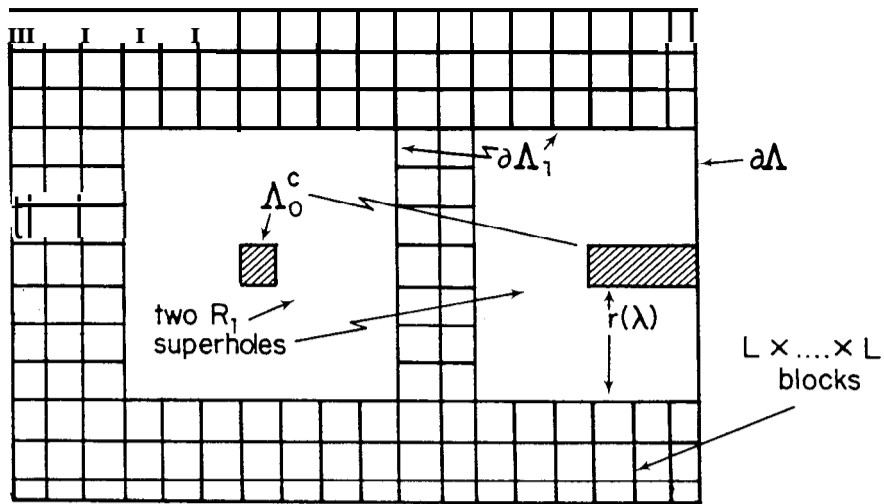


Fig. 2.

decomposition of  $\Lambda_1^c$  into superholes. The reason for organising the expansion this way can be understood by first simply pretending the  $\Xi^*$  factor is absent. Then a superhole  $R_1$ , containing  $R_0^{(1)}, R_0^{(2)}, \dots, R_0^{(N)}$  holes, including the ratios from the above expansion, has the weight

$$\int d^R \phi e^{-S(R)} \zeta_R Z_0^*(R_1 \setminus R) / Z_0(R_1), \quad (5.15)$$

where  $R$  is the union of holes in  $R_1$ . This is less than the same expression with  $S \rightarrow S_0$  and by definition of  $Z_0^*$  the  $S_0$  expression equals

$$\int d^{R_1} \phi e^{-S_0(R_1)} \zeta_R / Z_0(R_1),$$

which is exactly what we need to apply (5.14), the estimate that forces holes to be rare.

For future reference we define

$$\rho(R, R_1, \phi) := e^{-S(R)} \zeta_R Z_0^*(R_1 \setminus R) / Z_0(R_1). \quad (5.16)$$

This will determine the activity of a superhole  $R_1$  of “species”  $(R, \phi \upharpoonright_R)$ . We have to consider  $\phi \upharpoonright_R$  as a parameter for a superhole because  $Z_0^*$  and  $\Xi^*$  depend on it.

The expansion for  $\Xi^*$  is the cluster expansion (eqs. (5.2) and (5.3)). We substitute these expansions into  $Z$  and divide both sides by  $Z(\mathbf{A})$  and

$$\Xi_0(\Lambda) := \prod_{x' \in \Lambda} \Xi_0(x'),$$

which gives us an expansion for  $\tilde{Z}$  defined by

$$\tilde{Z} := Z/Z_0(\Lambda)\Xi_0(\Lambda). \quad (5.17)$$

We expand  $\exp(\Sigma(W^* - W)(r))$  and unite the total expansion for  $\tilde{Z}$  into one polymer system,

$$\tilde{Z} = \sum_0^\infty \frac{1}{M!} \sum_{C_1, \dots, C_M \subset \Lambda} \prod_{i=1}^M \bar{K}(C_i), \quad (5.18)$$

using the method described just before this example. Thus  $C$  is summed over unions of  $L \times \dots \times L$  blocks, which are connected in the sense that any block in  $C$  can be reached from any other via a path which traverses blocks which touch at faces, . . . , edges, or corners.  $\bar{K}$  is given by

$$\begin{aligned} \bar{K}(C) = \sum_{R_1, \mathbf{A}, \Gamma} \mathbf{1} \left( \sum_{R \subset R_1} \int d^R \phi \rho(R_1, R, \phi) \Xi^*(R_1 \setminus R) / \Xi_0(R_1) \right) \\ \times \prod_{\Delta \in \mathbf{A}} K^*(\Delta) \prod_{\Gamma \in \Gamma} (W^* - W)(\Gamma), \end{aligned} \quad (5.19)$$

where the sums over  $R_1, \mathbf{A}, \Gamma$  are to be interpreted as

$$\sum_{\Gamma} (\cdot) \equiv \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\Gamma_1, \Gamma_2, \dots, \Gamma_N} (\cdot).$$

The  $\mathbf{A}$ 's are what we used to call  $\Gamma$ 's in the expansion (5.2). The sums over  $R_1, \mathbf{A}, \Gamma$  obeys the compatibility condition that the blocks linked by all the  $\mathbf{A}$ 's and  $\Gamma$ 's and the  $R_1$ 's are contained in  $C$  and cover  $C$ .

We now would like to prove a diluteness condition, e.g (3. 10a), on  $\bar{K}(C)$  so that we can take the log of  $Z$  (or  $\tilde{Z}$ ). Such a condition is implied by

$$K(C) \leq \exp(-\bar{c}|C'|), \quad (5.20)$$

with  $\bar{c}$  sufficiently large. We will show that we can make  $\bar{c}$  as large as we like by taking  $L$  large followed by  $\mathbf{A}$  small.

According to the discussion given before this example, we define a

partition function  $Z_\alpha(C)$  by relaxing the constraint, in  $K(C)$ , that  $C$  be covered to merely that each  $R_1, \mathbf{A}, \Gamma$  lie inside  $C$  and we change the activities according to

$$\begin{aligned}\rho(R_1, \mathbf{R}, \phi) &\rightsquigarrow \rho_\alpha(R_1, R, \phi) := \rho(R_1, R, \phi) e^{\alpha|R_1|} \\ K^*(\Delta) &\rightsquigarrow K_\alpha(\Delta) := |K^*(\Delta)| e^{\alpha|\Delta|} \\ (\mathbf{W}^* - \mathbf{W})(\Gamma) &\rightsquigarrow W_\alpha(\Gamma) := |(\mathbf{W}^* - \mathbf{W})(\Gamma)| e^{\alpha|\Gamma|}.\end{aligned}$$

Then (5.20) is implied if we can show that for any  $\alpha$ , if  $L$  is large enough and  $A$  small enough

$$\mathbf{Z}_\alpha(C) \leq e^{|C'|}. \quad (5.21)$$

**Proof of (5.21) and thus (5.20).** The sum over  $\Gamma$ 's in  $Z_\alpha(C)$  is, for  $L$  large, less than  $\exp(\sum_{\Gamma \subset C} W_\alpha(\Gamma))$  which is less than

$$\exp\left(\text{const.} \left| \bigcup_{R_1 \in \mathbf{R}_1} R'_1 \right| L^{d-1} e^{\alpha d}\right), \quad (5.22)$$

because  $(\mathbf{W}^* - \mathbf{W})(\Gamma)$  vanishes unless  $\Gamma$  touches an  $R_1$ , and we have used (5.6). This means that the sum over  $\Gamma$ 's can be absorbed into an extra  $\exp(c(L)|R'_1|)$  factor per superhole.

For summing over  $\mathbf{A}$ 's we have to consider  $\mathbf{A}$ 's which link only  $L \times \cdots \times L$  blocks separately, i.e. we first discuss  $\mathbf{A}$ 's which do not link  $R_1$ 's. The principle we use is

$$\sum_{\Delta \text{ in } C} \prod_{\Delta \in \Delta} K_\alpha(\Delta) \leq \exp\left(|C'| \sup_{\mathbf{x}' \text{ A links } \mathbf{x}'} \sum K_\alpha(\Delta)\right).$$

The estimate (5.11) and remarks just below it tell us that if  $L$  is large and  $A$  is small then

$$K_\alpha(\Delta) \leq A e^{-c_1|\Delta|},$$

with  $c_1 \gg 1$ . This implies

$$\sup_{\mathbf{x}' \text{ A links } \mathbf{x}'} \sum K_\alpha(\Delta) \leq c_2 \lambda$$

and so

$$\sum_{\Delta \text{ in } C} \prod_{\Delta \in \Delta} \mathbf{K}^*(\mathbf{A}) \leq e^{c_2 \lambda |C'|}. \quad (5.23)$$

Notice the first estimate is valid with or without the condition  $\mathbf{A} \subset C$ . This will be useful in the next paragraph.

Now we consider the final case

$$\sum_{\Delta \text{ in } C} \sum_{\Delta \in \Delta} K_\alpha(\Delta),$$

where every  $\mathbf{A}$  in  $\mathbf{A}$  links at least one  $R_1$  factor. We shall see that this is less than

$$e^{c|\cup_{R_1 \in \mathbf{R}_1} R_1|} \prod_{R_1 \in \mathbf{R}_1} (\Xi(R_1 \setminus R))^{-1}. \quad (5.24)$$

The combinatoric aspect is the same as what we have just done because if we remove the  $R_1$  block(s) from each  $\mathbf{A}$  the remaining part of each  $\mathbf{A}$  is several connected graphs of the kind we were just discussing except that they must all touch an  $R_1$ , so we can perform the sum over  $\mathbf{A}$  by summing over sets of arbitrary graphs (on  $\mathbf{L} \times \dots \times \mathbf{L}$  blocks) each of which touch  $\cup_{R_1 \in \mathbf{R}_1} R_1$ .

Furthermore we have from (5.11), the combinatoric weight  $\exp(-c_1|\Delta|)$  per  $\mathbf{A}$  needed to do this sum. Summing such a weight produces part of the  $\exp(c|\cup_{R_1 \in \mathbf{R}_1} R_1|)$  in (5.24), cf. (5.23). The  $\exp(-c_1|\Delta|)$  can no longer completely dominate the  $c_2|X(\Gamma)|$  in (5.11) because the  $R_1$  blocks are arbitrarily large as  $A \rightarrow 0$  so this produces the rest of the  $\exp(c|\cup_{R_1 \in \mathbf{R}_1} R_1|)$  in (5.24).

The formula for  $K_\alpha$  is

$$K_\alpha(\Delta) = e^{\alpha|\Delta|} \left| \int ds_\Delta \partial^\Delta \Xi^*(X(\Delta), s_\Delta) / \Xi^*(X(\Delta), \mathbf{0}) \right|.$$

The  $\Xi^*(X(\Delta), 0)$  factors into  $\Xi^*(R_1 \setminus R)$ 's which consequently appear in (5.24) and  $\mathbf{L} \times \dots \times \mathbf{L}$  blocks  $\Xi(x')$ 's which as  $A \rightarrow 0$  are  $1 + O(A)$  and can be absorbed into the  $\exp(-c_1|\Delta|)$  in (5.11). Thus (5.24) is valid.

The  $\Xi^*$  factors in (5.24) cancel against the same factors in the superhole activities, see (5.19). Thus we find that after summing over all  $\mathbf{A}$ 's and  $\Gamma$ 's we are left with

$$Z_\alpha(C) \leq \sum_{R \text{ in } \mathbf{C}} \prod_{R_1 \in R_1} \left( \sum_{RCR_1} \int d^R \phi \rho(R_1, R, \phi) \Xi_0^{-1}(R_1) \right. \\ \left. \times \exp(c(\alpha, L)|R_1'|) \right) e^{c\lambda|C'|}.$$

The quantity  $\Xi_0(R_1)$  factors across all  $L \times \dots \times L$  blocks each of which contributes  $1 + O(A)$ . Thus by our previous remarks (5.14) and (5.15) and below, concerning the superholes, if  $A$  is small enough,

$$Z_\alpha(C) \leq \exp|C'|.$$

End of proof of (5.20) and (5.21).

## 6. Expansions for one-dimensional structures: self-avoiding random walk [21]

Suppose we have a set of objects naturally labelled by  $s = 0, 1, 2, \dots, |I|$ , i.e., a finite interval  $I \subset \mathbb{Z}^+$  (e.g., successive sites  $\omega(s)$ ,  $s = 0, 1, 2, \dots$ , visited by a walk  $\omega$  on the lattice  $\mathbb{Z}^d$ ). To any such a set of objects we associate a Gibbs factor of the form

$$\psi(I) := \exp\left(-\sum_{\substack{s < t \\ st \in I}} v_{st}\right),$$

but in this case it is natural to define connected parts differently, namely by solving recursively  $\psi_c(I) = \psi(I)$  if  $I = 1$ , otherwise

$$\psi(I) = \sum_n \frac{1}{n!} \sum_{I_1, \dots, I_n} \psi_c(I_1) \cdots \psi_c(I_n) \quad (6.1)$$

where  $I_1, \dots, I_n$  are **intervals** with disjoint interiors which fill  $I$ .

In analogy to (3.1) it is easy to show that for  $|I| > 1$ ,

$$\psi_c(I) = \sum_{\mathbf{G}} \prod_{st \in \mathbf{G}} (e^{-v_{st}} - 1), \quad (6.2)$$

where  $\mathbf{G}$  is summed over all graphs, connected in the sense that: every point strictly inside  $I$  is covered by a bond  $st \in \mathbf{G}$  and the end-points belong to bonds in  $\mathbf{G}$ . A point  $u$  in  $I$  is covered by a bond  $st$  in  $\mathbf{G}$  if



Fig. 3.

$u \in (s, t)$ . In this case a minimally connected graph  $\mathbf{L}$  on an interval  $[S, T]$  looks like fig. 3. This is a graph with the property that if any bond is removed it becomes disconnected. We say that  $G$  is a **lace** if it has this property of becoming disconnected when any line is removed. The laces are the analogous of tree diagrams in our previous sections and they play the same role.

**Theorem 6.1.** *To each lace  $L$  on an interval  $I$  is associated a set of bonds  $\mathcal{C}(L)$  such that*

$$\psi_c(I) = \sum_{L \text{ on } I} \prod_{st \in L} (e^{-v_{st}} - 1) \prod_{st \in \mathcal{C}(L)} e^{-v_{st}}.$$

$\mathcal{C}(L)$  contains (among others) all bonds  $uv$  with  $[u, v] \subseteq [s, t]$  for some  $st \in L$ .

**Proof.** Given a connected graph  $G$  on  $I = [0, T]$  we define a distinguished subgraph  $\mathbf{L}(G)$  whose bonds  $s_1 t_1, s_2 t_2, \dots$ , are determined by

$$t_1 := \max\{t: \mathbf{O}t \in G\}, \quad t_{i+1} := \max\{t: st \in G, (s, t) \cap (0, t_i) \neq \emptyset\},$$

$$s_1 := 0, \quad s_i := \min\{s: st_i \in G\}.$$

By trying this recipe on a few graphs it will become clear that  $\mathbf{L}(G)$  is always a lace. Now we resum connected graphs by writing, using (6.2),

$$\begin{aligned} \psi_c(I) &= \sum_{G \text{ on } I} \prod_{st \in G} (e^{-v_{st}} - 1) \\ &= \sum_{L \text{ on } I} \prod_{st \in L} (e^{-v_{st}} - 1) \left\{ \sum_{G: \mathbf{L}(G)=L} \prod_{st \in G \setminus L} (e^{-v_{st}} - 1) \right\}. \end{aligned}$$

Define  $\mathcal{C}(L) = G_m \setminus L$  where  $G_m :=$  maximal graph  $G$  such that  $\mathbf{L}(G) = L$ , then the factor in braces equals

$$\prod_{st \in \mathcal{C}(L)} e^{-v_{st}}.$$

To see this write  $e^{-v_{st}} = (e^{-v_{st}} - 1) + 1$  and expand out the product over  $st \in \mathcal{C}(L)$ . End of proof.

This theorem immediately implies that if  $v$  is **repulsive**

$$|\psi_c(I)| \leq \sum_{L \text{ on } I} \prod_{st \in L} |e^{-v_{st}} - 1|, \quad (6.3)$$

which is a huge improvement over (6.2). However it is possible to do still better. A lace  $L = \langle s_i t_i : i = 1, 2, \dots, N \rangle$  on  $I$  determines a partition of  $I$  into subintervals  $I_j$  with disjoint interiors: write out the time  $s_1, t_1, s_2, t_2, \dots, s_n, t_n$  in their natural order and then the closed intervals between them are the subintervals  $J_j, j = 1, \dots, 2N - 1$ . We write  $J < L$  to denote a subinterval of  $I$  obtained from  $L$  in this way.

From theorem 6.1 it is immediate that for repulsive interactions

$$|\psi_c(I)| \leq \sum_{L \text{ on } I} \prod_{st \in L} |e^{-v_{st}} - 1| \prod_{J < L} \psi(J). \quad (6.4)$$

This inequality allows a recursive high-temperature expansion: roughly speaking, a crude estimate (such as  $\exp(-v) \leq 1$ ) on the  $\psi$  inside a partition function gives via (6.4) an estimate on an object  $\Pi$  involving  $\psi_c$  which then yields a better estimate on the partition function etc. Notice that (6.4) involves the partition function for subintervals  $J < I$  because of the factors  $\Psi(J)$ . We will explain this in more detail below.

**Self-avoiding random walks.** We define a sum over self-repelling walks from 0 to  $x$

$$N(x, T) := \sum_{\substack{\omega: 0 \rightarrow x \\ |\omega| = T}} \left( \frac{1}{2d} \right)^{|\omega|} \psi([0, T], \omega).$$

The Gibbs factor  $\psi$  assigns a factor  $\exp(-\beta)$  for each self intersection **within a memory**  $\tau$ :

$$\begin{aligned} \psi([0, T], \omega) &:= \exp \left[ - \sum_{st, 0 \leq s < t \leq T} \beta(t-s) \delta(\omega(t) - \omega(s)) \right], \\ \beta(s) &= \beta \text{ i f } |s| \leq \tau, \\ &= 0 \text{ otherwise.} \end{aligned}$$



It is also convenient to have various Fourier and Laplace transforms:

$$\begin{aligned} \mathbf{N}(\mathbf{k}, \mathbf{T}) &:= \sum_{x \in \mathbb{Z}^d} N(x, \mathbf{T}) e^{i\mathbf{k} \cdot x}, \\ N(x, \mathbf{z}) &:= \sum_{T=0}^{\infty} z^T N(x, \mathbf{T}), \\ N(k, \mathbf{z}) &:= \sum_{T=0}^{\infty} z^T \sum_{x \in \mathbb{Z}^d} N(x, \mathbf{T}) e^{i\mathbf{k} \cdot x}. \end{aligned}$$

If  $\beta$  (or  $\tau$ ) = 0 so that there is no interaction, then

$$N(k, \mathbf{T}) = D(k)^T, \quad \text{with } \mathbf{D}(\mathbf{k}) := \frac{1}{d} \sum_{j=1}^d \cos k_j,$$

$$N(k, z) = (1 - zD(k))^{-1}.$$

The object is to calculate corrections to these exact results caused by turning on the interaction, in particular the asymptotic behavior of  $\mathbf{N}(\mathbf{k}, \mathbf{T})$  as  $T \rightarrow \infty$ . The exponential decay in  $\mathbf{T}$  will be governed by  $\mathbf{r}(\mathbf{k}) :=$  radius of convergence of  $N(\mathbf{k}, z)$ . We set  $r := r(\mathbf{0})$ .  $\mathbf{r}(\mathbf{k})$  also depends on  $\beta$  and  $\tau$ . Note that  $r \geq r(\beta = \mathbf{0}) = 1$ .

To each of the quantities  $N(x, \mathbf{T})$ ,  $N(k, \mathbf{T})$ ,  $N(k, z)$ , is a counterpart  $\Pi(x, \mathbf{T})$ ,  $\Pi(k, \mathbf{T})$ ,  $\Pi(k, z)$  defined by replacing  $\psi$  by  $\psi_c$  in the definitions. The cluster expansion (6.1) can be inserted into  $N(k, z)$  and, after a short calculation, yields

$$\mathbf{N}(\mathbf{k}, z) = [1 - z\mathbf{D}(\mathbf{k}) - \Pi(k, z)]^{-1}, \quad (6.5)$$

so the  $\beta = 0$   $N(k, z)$  is modified by  $\Pi$ . The algebra corresponding to (6.1) is no longer taking the log as in sect. 2 because the notion of connectedness has changed. (It is in fact one-particle-irreducibility in field theory.) It is possible to prove the following theorem [21].

**Theorem 6.2.** *Suppose  $d \geq 5$ ,  $\tau = \infty$  and  $\beta$  is small, then*

$$\mathbf{N}(\mathbf{k}, \mathbf{T}) \sim r^{-T} e^{-\frac{1}{2}\Delta T k^2}, \quad \text{as } T \rightarrow \infty.$$

**The precise statement is**

$$N(k/\sqrt{s}, sT) / N(0, sT) \xrightarrow{s \rightarrow \infty} e^{-\frac{1}{2}\Delta T k^2}$$

**uniformly on compact sets in  $T$  and  $k$ .  $A$  is a constant.**

This theorem says that at long distances the effect of the interaction is merely to renormalise the parameters: killing rate and diffusion constant from their bare ( $\beta = 0$ ) values. We will not give a complete proof of this theorem but we will explain how it is possible to learn about  $N(\mathbf{k}, T)$  for arbitrarily large  $\tau$  and  $T$  provided  $T \gg \tau$ . To begin with we prove

**Proposition 6.3.** Let  $\tau < \infty$  and  $\beta$  small depending on  $\tau$ . Then  $\Pi(k, \mathbf{z})$  is analytic in  $\mathbb{D}_3$  where

$$\mathbb{D}_a := \{z: |z| \leq 1 + a/\tau\}.$$

For  $\mathbf{k}$  small,  $k^2 \leq 2d/\tau$ ,  $z = \mathbf{r}(\mathbf{k})$  is the unique solution, inside  $\mathbb{D}_3$ , of

$$1 - zD(\mathbf{k}) - \Pi(k, \mathbf{z}) = 0,$$

$\mathbf{r}(\mathbf{k})$  is real analytic in  $\mathbf{k}$  and lies inside  $\mathbb{D}_2$ ,  $|\mathbf{r}(\mathbf{k})| \geq r$ . Furthermore

$$N(k, T) = \left[ D(k) + \frac{\partial}{\partial z} \Pi(k, r(k)) \right]^{-1} r(k)^{-T-1} + \mathbf{O}\left(1 + \frac{3}{\tau}\right)^{-T-1}.$$

This is a statement about the asymptotics as  $T \rightarrow \infty$  of  $N(k, T)$  for small  $\mathbf{k}$ . It is also possible to obtain some information about large  $\mathbf{k}$ .

**Proposition 6.4.** Under the same conditions on  $\tau$  and  $\beta$  as above, if  $k^2 \geq 2d/\tau$  and  $(\mathbf{k} - \boldsymbol{\pi})^2 \geq 2d/\tau$ ,  $\boldsymbol{\pi} = (\pi, \dots, \pi)$ , then  $|N(k, T)| \leq O((\text{radius of } \mathbb{D}_{1/2})^{-T})$  (as  $T \rightarrow \infty$ ).

**Proof of proposition 6.3** (Details in [21]). The main step is to prove that  $\Pi$  is analytic in  $\mathbb{D}_a$  if  $\beta$  is small.

*Analyticity of  $\Pi$ .* By the definition of  $\Pi(k, \mathbf{z})$ ,

$$|\Pi(k, z)| \leq \sum_{x \in \mathbb{Z}^d} \sum_{T=1}^{\infty} |z|^T |\Pi(x, T)|.$$

We substitute in  $\Pi(x, T)$  the bound (6.4). No lace  $L$  with number of lines  $N$  smaller than  $T/\tau$  can contribute because each line  $st$  in  $L$  satisfies  $|t - s| \leq \tau$  and  $L$  must be connected on  $[0, T]$ . Consequently for  $z$  and  $p$  such that  $|z| \geq p$

$$|z|^T = ((|z|/\rho)^\tau)^{T/\tau} \rho^T \leq (|z|/\rho)^{\tau N} \rho^T.$$

[This is an essential point in the argument. It is what we need to compensate the exponential growth of  $|z|^T$  by the  $N$  factors  $(\exp(-\beta) - 1)$  prescribed by  $L$ .]

Each bond  $st \in L$  prescribes a factor  $\exp(-\beta\delta_{st}) - 1$  which equals  $(\exp(-\beta) - 1)\delta_{st}$ . The  $\delta$  functions force  $\omega(s) = \omega(t)$  for each  $st \in L$  so to each  $L$  is associated a topology  $\mathcal{G}$  of intersections. The first few  $\mathcal{G}$ 's are shown in fig. 4. By splitting the walks into subwalks connecting the vertices of  $\mathcal{G}$  and summing over each subwalk and its duration we see the laces correspond to Feynman diagrams and

$$|\Pi(k, z)| \leq \sum_{N=1}^{\infty} [ (|z|/\rho)^\tau |e^\beta - 1| ]^N \sum_{x_2, \dots, x_N} \prod_{ij \in \mathcal{G}_N} N(x_i - x_j, p), \quad (6.6)$$

where  $|z| \geq p$ ,  $N(x, p) = N(x, z = p)$  and  $\mathcal{G}_1, \mathcal{G}_2, \dots$  are the Feynman diagrams in fig. 4. Analogous bounds hold for  $k$  and  $z$  derivatives of  $\Pi$ .

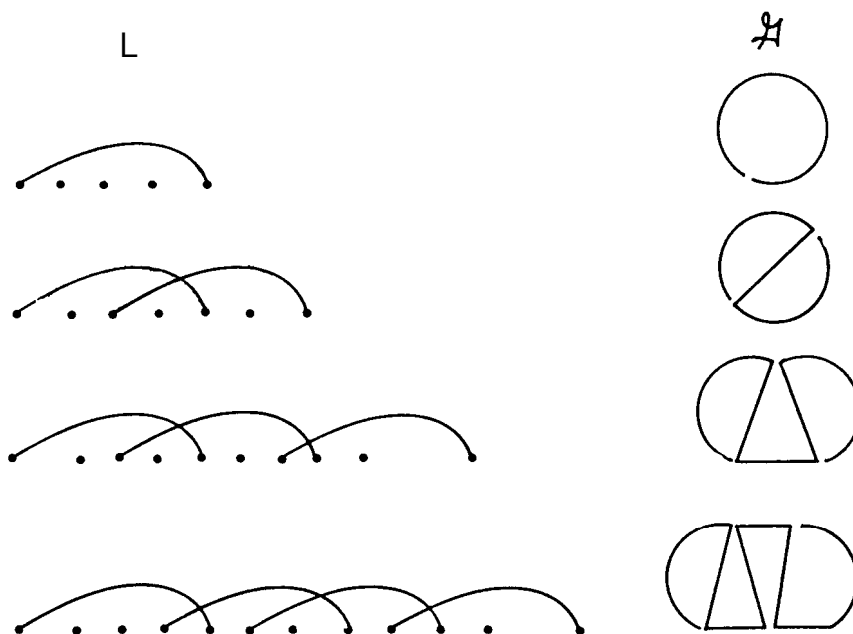


Fig. 4.

The  $\psi(I)$ 's in (6.4) give the factors  $N(x, \mathbf{p})$  in this estimate. For the present proof it is enough to throw out all the self-avoidance in  $N(x, \mathbf{p})$  and set  $\mathbf{p} = 1$ . By the definition of  $N(x, z)$ ,

$$N(x, 1) \leq \sum_{\omega: 0 \rightarrow x} (1/2d)^{|\omega|} \leq \text{const.}/(1+|x|)^{d-2}, \quad (6.7)$$

i.e., we have related  $N(x, 1)$  to simple random walk and used its known large- $x$  asymptotics. This bound is good enough to substitute into (6.6), the bound for  $\Pi(k, z)$ , because all these graphs  $\mathcal{G}_N$  are infra-red convergent in  $d \geq 5$ .

Infra-red convergence implies [22] that each graph  $\mathcal{G}_N$  gives contribution bounded by  $C^N$  and so by (6.6)  $\Pi$  is analytic in  $\mathbb{D}_a$  and  $O(\beta)$  if

$$C e^a |e^{-\beta} - 1| < 1.$$

Instead of [22], one can also use lemma 6.5 (below) to bound the graphs. The use of  $d \geq 5$  can be avoided by taking  $\mathbf{p} < 1$  so that  $N(x, \mathbf{p})$  has exponential decay. Then the domain of analyticity is  $z/\rho \in \mathbb{D}_a$  but this is still sufficient to prove analyticity in  $\mathbb{D}_3$  since we can play with  $a, \beta$ .

**Properties of  $r(\mathbf{k})$ .** By taking  $\beta$  small we can, since  $\Pi = O(\beta)$ , achieve

$$|\Pi(k, z)| < |1 - zD(k)|,$$

for  $k^2 \leq 2d/\tau$  and  $z \in \partial\mathbb{D}_3$ . By Rouché's theorem (I am indebted to the Chayes for lending me this theorem), the number of zeros of  $1 - zD - \Pi$  within  $\mathbb{D}_3$  is equal to the number of zeros of  $1 - zD$  in  $\mathbb{D}_3$ , i.e., one. Thus there is a unique zero and of course it must be  $z = r(\mathbf{k})$  since this is the distance to the nearest singularity of  $N(k, z) = (1 - zD - \Pi)^{-1}$ . The zero is real because it solves an equation with real coefficients and is unique. It equals  $+r(\mathbf{k})$  and is smooth in  $\mathbf{k}$ ,  $|k|^2 \leq 2d/\tau$  by the implicit function theorem.

**Asymptotic behaviour of  $N(k, T)$ .** We use the Cauchy formula

$$N(k, T) = \frac{1}{2\pi i} \oint (1 - zD(\mathbf{k}) - \Pi(k, z))^{-1} \frac{dz}{z^{T+1}}$$

and deform the contours to  $\partial\mathbb{D}_3$  picking up the residue of the pole at

$z = \mathbf{r}(\mathbf{k})$ , which is the first term, and a contribution from the contour around  $\partial\mathbb{D}_3$  which is the second term in the asserted asymptotic behaviour as  $T \rightarrow \infty$ . End of proof of proposition 6.3.

**Proof of proposition 6.4.** We express  $\mathbf{N}(\mathbf{k}, \mathbf{T})$  by the Cauchy formula as above and deform the contour to  $\partial\mathbb{D}_{1/2}$ . Inside  $\mathbb{D}_{1/2}$ ,  $1 - zD$  has no zeros for  $k^2 \geq 2d/\tau$  and  $(\mathbf{k} - \boldsymbol{\pi})^2 \geq 2d/\tau$ , furthermore by taking  $\beta$  small  $|\Pi| < |1 - zD|$  on  $\partial\mathbb{D}_{1/2}$  so  $1 - zD - \Pi$  also has no zeros. Consequently, there is no residue contribution as in proposition 6.3 and  $N(k, \mathbf{T})$  is bounded by the contribution of the contour around  $\partial\mathbb{D}_{1/2}$ . This gives the result of proposition 6.4. The condition  $(\mathbf{k} - \boldsymbol{\pi})^2 \geq 2d/\tau$  is there because  $F := 1 - zD - \Pi$  changes to  $F$  under  $k \rightarrow k - \boldsymbol{\pi}$  and  $z \rightarrow \bar{z}$ .

Now we turn to the problem of proving the same sort of result as in proposition 6.3 but with no restriction on the size of  $\tau$  except that  $\mathbf{T} \gg \tau$ .

**Renormalisation group argument.** Let  $\tau' \gg \tau$ , indeed we will describe step one in an induction on the sequence  $\tau_n = 2^{n\alpha}$ . This will not be a complete argument. We refer the reader to [21] for more details.

We repeat the proof of proposition 6.3 with  $\tau$  replaced by  $\tau'$  but in using (6.6) to estimate  $\Pi'$  (primes mean  $\tau$  is replaced by  $\tau'$ ) we take advantage of having learnt about  $N(x, \mathbf{T})$  on scales  $\leq \tau$ . This is done by choosing  $\mathbf{p} = r$  (which is greater than 1) and proving that

$$“ N(x, T) \leq C_1 r^{-T} T^{-d/2} e^{-\frac{1}{2}C_2|x|^2/T} ” \quad (6.8)$$

which implies

$$“ N(x, z = r) \leq \frac{C}{|x|^{d-2}} ” \quad (6.8')$$

$C$  is used to denote constants with no  $\tau$  or  $\beta$  dependence. The result is:

**Propositions 6.3' and 6.4'.** The same as propositions 6.3 and 6.4 but with  $\tau$  replaced by  $\tau'$ ,  $\mathbb{D}_a$  replaced by

$$\mathbb{D}'_a := \left\{ z: |z|/r \leq 1 + \frac{a}{\tau'} \right\},$$

and  $\beta$  **unchanged**.

Notice that  $\mathbb{D}_a$  can be thought of as

$$\mathbb{D}_a = \left\{ z: |z| / r^{(\tau=0)} \leq 1 + \frac{a}{\tau} \right\},$$

and since  $\tau = 0$  is the predecessor scale to  $\tau$  the definitions of  $\mathbb{D}$  and  $\mathbb{D}'$  are consistent.

It is clear from (6.6) how  $\mathbb{D}$  gets replaced by  $\mathbb{D}'$ , once one has (6.8'). We will now explain why  $\beta$  need not be changed assuming we have (6.8'). The only reason why we might have to take  $\beta$  smaller is that  $\mathbb{D}'_2$  might not any longer contain  $z = r'(k)$  which should also be the unique solution to  $1 - zD - \Pi' = 0$ . We need to check the conditions for Rouché's theorem:

$$|\Pi' - \Pi| < |1 - zD - \Pi|$$

on  $\mathbb{D}_a$ ,  $a = \frac{1}{2}, 3$ , i.e., we want  $1 - zD - \Pi$  and  $1 - zD - \Pi'$  to have the same number of zeros in  $\mathbb{D}'_a$ ,  $a = \frac{1}{2}, 3$ . (Calculations of the type we are about to demonstrate show that  $\mathbb{D}' \subset \mathbb{D}$ .) We estimate the left-hand side by a version of (6.6) together with (6.8). For example, the first lace contribution to  $\Pi' - \Pi$  is, according to fig. 4, (6.6) and (6.8),

$$|\Pi' - \Pi|_{z \in \partial \mathbb{D}'_a}^{(\text{first graphs})} \sim \left( \sum_{T \geq \tau} T^{-d/2} \right) |e^{-\beta} - 1| \leq \tau^{-d/2+1} |e^{-\beta} - 1|,$$

we are leaving out constants independent of  $\beta$  and  $\tau$ . On the other hand by considering the distance between  $r$  and  $\partial \mathbb{D}'_a$ , which is  $O((\tau')^{-1})$ ,

$$|1 - zD - \Pi| \sim (\tau')^{-1},$$

for  $k^2 \leq 2d/\tau'$  and  $z \in \mathbb{D}'_a$ . This follows by showing that the  $z$  derivative of  $1 - zD - \Pi$  is close to 1. By comparing the two we see that we will not need to decrease  $\beta$  some more in-order to achieve the conditions of Rouché's theorem if  $d \geq 5$  and  $\tau' \leq \tau^\alpha$ ,  $\alpha \geq 1$ .

*The estimate* (6.8). In fact estimates (6.8) are not proved! (hence the quotations) because it is difficult to transform data on  $N(k, \mathbf{T})$  as obtained in proposition 6.3 into such an estimate in  $x$ -space. Instead we prove that (6.8) holds in the sense of  $L_p$  norms ( $p = 1, \infty$ ) of both sides. This is just as good because of

**Lemma 6.5.** ([21] sect. 5.)

$$\left| \sum_{x_2, \dots, x_N} \prod_{ij \in \mathcal{G}_N} N(x_i - x_j, \rho) \right| \leq \left( \prod_{ij \in \mathcal{G}_N \setminus \text{one line}} \|N(\rho)\|_2 \right) \|N(\rho)\|_\infty.$$

We estimate  $\|N\|_p$  at  $z = \rho = r$  by

$$\|N(r)\|_p \leq \sum_{T=0}^{\infty} r^T \|N(\cdot, T)\|_p \leq \sum_{T=0}^{\infty} r^T \|N^{(\tau \rightarrow \sigma(T))}(\cdot, T)\|_p,$$

where  $a(T)$  equals  $\tau$  if  $T \gg \tau$ , otherwise is smaller. This step is necessary to avoid trouble from the second term in the asymptotic behavior of  $N(k, T)$ . (We need  $T \gg \text{memory}$ .) Lastly, we use the Hausdorff inequality

$$\|N\|_p^{(x \text{ space})} \leq \|N\|_q^{(k \text{ space})},$$

$1/p + 1/q = 1$ ,  $p \leq 2$  together with propositions 6.3 and 6.4.

Proposition 6.3 says

$$\begin{aligned} N(k, \mathbf{T}) &\sim r(k)^{-T}, \quad \text{for } \mathbf{T} \gg \tau, \\ &\sim (r + \frac{1}{2}(\partial_k^2 r)(0)k^2)^{-T}, \end{aligned}$$

for  $\mathbf{k}$  small (and an analogous statement for  $\mathbf{k} - \boldsymbol{\pi}$  small),  $k^2 \leq 2d/\tau$ ,

$$\sim r^{-T} (1 + \frac{1}{2} \Delta k^2)^{-T},$$

where  $\mathbf{A} = (\partial_k^2 r)(0)/r$

$$\leq r^{-T} e^{-\frac{1}{2} C T k^2} \quad \text{with } C \simeq \mathbf{A}.$$

Proposition 6.4 shows that if  $\mathbf{T} \gg \tau$  the contribution to the  $L_q$  integral is predominantly from the small  $\mathbf{k}$  or  $\mathbf{k} - \boldsymbol{\pi}$  region. The above Gaussian formula leads to the  $L_q$  form of (6.8). If  $\mathbf{T}$  is not much larger than  $\tau$  then we replace  $\tau$  by  $\sigma' \ll \mathbf{T}$  and by induction we already have propositions 6.3 and 6.4 for this scale. We also know that the difference between  $r^{(\sigma)}$  and  $r^{(\tau)}$  **goes** as  $\sigma^{-d/2+1}$  (see the Rouché argument above) and for  $d \geq 5$  this is small enough to obtain still the  $L_q$  form of (6.8). See the end of sect. 5 in ref. [21] for more details. (It is not difficult in fact.)

By these arguments together with some care over whether symbols like  $O(\cdot, \cdot)$  have a uniform meaning as  $\tau$  is raised, it is possible to repeatedly replace propositions 6.3 and 6.4 by their primed versions for a scale of  $\tau$ 's,  $\tau_n \sim 2^{na}$  until any desired scale is reached with just one initial choice of  $\beta$ .

The successive diffusion constants remain within  $O(\beta)$  of 1 because the  $\mathcal{G}_N$  diagrams corresponding to  $\partial_z^k r(0)/r = \mathbf{A}$  are infra-red convergent for  $d \geq 5$ . End of discussion.

### Appendix A. Formal series for $\log Z$ and eq. (3.1)

**Proof of (2.3b).** We substitute the definition of connected parts (2.2) into the partition function (2.3a) and obtain

$$Z = \sum_N \frac{1}{N!} \sum_{\pi \in \text{partitions of } \{1, 2, \dots, N\}} \prod_{Y \in \pi} \int d^Y \mu \psi_c(Y).$$

Noting that the  $d^Y \mu$  integral only depends on  $|Y|$  we define  $f(n)$  by

$$f(\mathbf{n}) := \int d^Y \mu \psi_c(Y) \text{ with } |Y| := \mathbf{n}$$

and sum over all partitions  $\pi \equiv \{Y_1, \dots, Y_M\}$ , with  $|Y_i| = n_i$  fixed,  $\sum n_i = N$ ,  $n_i \geq 1$ ,  $M$  fixed,

$$\begin{aligned} Z &= \sum_N \frac{1}{N!} \sum_M \frac{1}{M!} \sum_{n_1, \dots, n_M} \frac{N!}{n_1! \cdots n_M!} \prod_{i=1}^M f(n_i) \\ &= c \frac{1}{M!} \sum_{n_1, \dots, n_M} \frac{1}{n_1! \cdots n_M!} \prod_{i=1}^M f(n_i) = \exp\left(\sum_n \frac{1}{n!} f(n)\right), \end{aligned}$$

which proves (2.3b). ■

**Proof of (3.1).** We expand  $\psi(X)$  out in graphs according to

$$\begin{aligned} \psi(X) &= \exp\left(-\frac{1}{2} \sum_{ij \in X} v_{ij}\right) \\ &= \prod_{i < j} (e^{-v_{ij}} - 1 + 1) \prod_{i \in X} e^{-\frac{1}{2} v_{ij}} \\ &= \sum_G \prod_{ij \in G} (e^{-v_{ij}} - 1) \prod_{ij \in X} e^{-\frac{1}{2} v_{ij}} \end{aligned} \tag{A.1}$$



$G$  is summed over all subsets of  $\{ij; \mathbf{i} < \mathbf{j}, ij \in X\}$  which may be visualised as graphs on  $X$ . Every  $G$  may be decomposed into a union of connected components:  $G = \bigcup G_i$  and each connected component  $G_i$  is a graph on some subset  $Y_i$  of  $X$ . The sets  $Y_1, \dots, Y_M$  define a partition  $\pi$  of  $X$ . Thus to every  $G$  is associated a unique partition  $\pi$ . We resum all the graphs  $G$  compatible with  $\pi$ :

$$\sum_G \prod_{ij \in G} (e^{-v_{ij}} - 1) = \sum_{\pi} \prod_{Y \in \pi} \left( \sum_{\substack{\mathbf{G} \text{ connected} \\ \text{on } Y}} \prod_{ij \in G} (e^{-v_{ij}} - 1) \right). \quad (\text{A.2})$$

We combine (A.1) and (A.2) into an equation for  $\psi(X)$  which we compare with (2.2), the definition of connected parts. (3.1) follows immediately. ■

### Appendix B. Proof of theorem 3.1

We will prove a ‘‘many-body potential’’ generalisation of theorem 3.1.

Suppose that the potential  $\mathbf{V}$  has the form

$$V(X) := \frac{1}{2!} \sum_{ij \in X} v_{ij} + \frac{1}{3!} \sum_{ijk \in X} v_{ijk} + \dots =: \sum_{b \subset X} \frac{V(b)}{|b|!}.$$

$\mathbf{b}$  is summed over finite sequences with values in  $X$ . The two-body case is obtained by taking  $\mathbf{u}(\mathbf{b}) = 0$  unless  $|b|$ , the length of the sequence, is two. Set operations with  $\mathbf{b}$  are performed with the set of **distinct** entries in  $\mathbf{b}$ . We number the elements in  $X$  1, 2, . . . and set

$$X_1 := \{1\}.$$

Introduce a potential  $W(X_1; s_1)$ ,  $s_1 \in [0, 1]$  by a convex decoupling across  $\partial X_1$ . This means

$$W(X_1; s_1) := \sum_{b \subset X} s_1(b) v(b) / |b|!,$$

$$\mathbf{s}(\mathbf{b}) := s_1 \quad \text{if } \mathbf{b} \text{ couples across } \partial X_1,$$

$$= 1 \quad \text{otherwise}.$$

$\mathbf{b}$  is said to **couple across**  $\partial Y$ ,  $\mathbf{b}$  or  $\mathbf{a}$   $Y$ , if  $\mathbf{b}$  contains objects in  $Y$  and the complement of  $Y$ . More generally, if

$$X_1 \subset X_2 \subset \cdots \subset X_r \subset X$$

is any sequence of strictly increasing subsets of  $X$  then we define

$$W(X_1, \dots, X_r; s_1, \dots, s_r) := \sum_{b \subset X} s_1(b) s_2(b) \cdots s_r(b) \frac{v(b)}{|b|!}$$

with  $s_i(b) = s_i$  if  $\mathbf{b}$  cr  $\partial X_i$ , otherwise  $s_i(b) = 1$ . This also is a convex decoupling. Now we attempt to decouple 1 from the rest of  $X$ . This means we expand  $\psi(X) \equiv \exp(-V(X))$  by writing

$$\begin{aligned} \psi(X) &= \int_0^1 ds_1 e^{\frac{\partial}{\partial s_1} W(X_1, s_1) + e^{-W(X_1, 0)}} \\ &= \sum_{b_1 \text{ cr } \partial X_1} (-v(b_1)) \int ds_1 e^{-W(X_1, s_1) + e^{-W(X_1, 0)}}. \end{aligned} \quad (\text{B.1})$$

We have expressed  $\psi(X)$  as the sum of two terms: in the second  $\{1\} \equiv X_1$  is decoupled from the rest of  $X$ , in the first a bond  $\mathbf{b}$  between 1 and elements in  $X$  is exhibited. We set  $X_2 = X_1 \cup b_1$  and substitute into the *first term* in (B. 1), under the sum over  $\mathbf{b}_1$ ,

$$e^{-W(X_1, s_1)} = \int ds_2 \frac{\partial}{\partial s_2} e^{-W(X_1, X_2, s_1, s_2) + e^{-W(X_1, X_2, s_1, 0)}}.$$

On performing  $\partial/\partial s_2$  a coupling across  $\partial X_2$  via a bond  $b_2$  will be exhibited allowing us to define  $X_3 = X_2 \cup b_2$  and so on. The result of continuing until  $X_r = X$  for some  $r$ , leaving decoupled terms alone, is an expansion for  $\psi(X)$  of the form

$$\psi(X) = \sum_{Y \ni 1} K(Y) \psi(X \setminus Y) \quad (\text{B.2})$$

where  $\psi(\emptyset) := 1$ ,

$$K(Y) := e^{-V(Y)} \text{ if } |Y| = 1,$$

otherwise

$$K(Y) := \sum_{r \geq 2} \sum_{X_2, \dots, X_{r-1}} \sum_{\text{Tree } Y} \prod_{b \in T} \left( -\frac{v(b)}{|b|!} \right) \\ \times \int ds_1 \cdots ds_{r-1} \prod_{b \in T} \left( \frac{\partial}{\partial s_{i(b)}} \prod_{i=1}^{r-1} s_i(b) \right) e^{-W(X, s, s_r=0)}, \quad (\text{B.3})$$

$$i(b) := \sup\{i: \mathbf{b} \text{ cr } \partial X_i\},$$

$$X := (X_1, \dots, X_r), \quad X_1 \subset X_2 \subset \cdots \subset X_r, \quad X_1 = \{1\}, X_r = Y,$$

and  $T$  is summed over all trees on  $Y$  consistent with  $X$ . In this many-body context, a graph consists of many-body bonds  $b_1, b_2, \dots, b_{r-1} \subset Y$  which connect the vertices, elements of  $Y$ , in a way analogous to the two-body case (for which each  $b_i$  has length = 2). A graph is a tree if it has the additional property that its vertices become disconnected if any bond is removed.

(B.2) can be iterated, and it becomes

$$\psi(X) = \sum_{\text{repartitions of } X} \prod_{Y \in \pi} K(Y).$$

We chose on ordering of  $X$  so that, for each  $Y \subset X$ , a first element in  $Y$  which will play the role of 1 is defined. By comparing this with (2.2) we see

$$\psi_c(Y) = K(Y).$$

By interchanging the sums over  $X$  and trees in (B.3) we see that (B.3) has the form<sup>†</sup>

$$\psi_c(Y) = \sum_T \prod_{b \in T} \left( -\frac{v(b)}{|b|!} \right) \int dp_T(s) e^{-V(s)}, \\ V(s) = \sum_{b \subset Y} s(b)v(b)/|b|!, \quad s = (\mathbf{s}(b)),$$

where  $dp_T(s)$  is supported on convex decouplings of  $Y$ . By the same arguments as presented in sect. 3, Federbush and Battle proved that  $dp_T$  is a probability measure in this many-body context. Thus a many-body version of theorem 3.1 is proven.

<sup>†</sup>These results contain factorials ( $|b|!$ ) not present in theorem 3.1 because bonds  $\mathbf{b}$  are **sequences** and are considered as distinct when the order of entries differ, unlike sets.

As in sect. 3 one can now prove some bounds analogous to corollary. **3.2.**

**Corollary B.1.**

(a) **If  $V$  is stable**

$$|\psi_c(X)| \leq \sum_{T \text{ on } X} \prod_{b \in T} \frac{|v(b)|}{|b|!}.$$

(b) **If  $V$  is repulsive without self energies**

$$|\psi_c(X)| \leq \sum_{T \text{ on } X} \prod_{b \in T} |e^{-v(b)/|b|!} - 1|/|b|!$$

These results are merely rewordings of results by Federbush and Battle in ref. [8].

**Problem (Hard).** These bounds are useful for lattice systems with many-body interactions but, unlike the two-body case, yield no information on continuum systems (unless they are converted to lattice systems). This is because there are too many tree graphs in the many-body context. Can an improvement be found which gives a direct proof of convergence of the Mayer expansion for continuum systems with many body potentials?

**Appendix C**

**Equation (3.10a):** the main point is how to estimate a contribution to  $\log Z$  from a fixed tree graph  $T$  on  $\{1, 2, \dots, N\}$  with incidence numbers  $d_1, \dots, d_N$ . We explain it by the example illustrated in fig. 5. The contribution of the tree graph displayed in fig. 5 is

$$\sum_{Y_1, \dots, Y_5 \subset A} K(Y_1) \cdots K(Y_5) \prod_{ij \in T} (e^{-v_{ij}} - 1) \quad (\text{C.1})$$

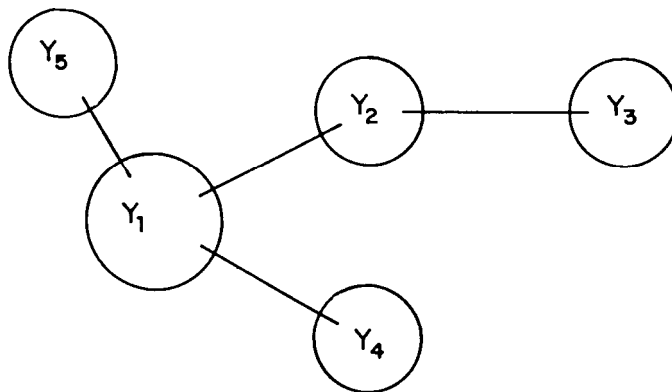


Fig. 5. Five polymers  $Y_1, \dots, Y_5$  linked by a tree graph on  $\{1, 2, \dots, 5\}$ .

where  $v_{ij} = \infty$  if  $Y_i \cap Y_j \neq \emptyset$ , 0 otherwise. Thus  $\exp(v_{ij}) - 1$  vanishes unless  $Y_i \cap Y_j = \emptyset$ . This means the polymers are forced to overlap in pairs as specified by lines  $ij$  in  $T$ . We bound (C.1) in absolute value by summing over polymers at the end of branches of the tree. For example, we sum over  $Y_3$  by holding fixed a point  $y_3$  in  $Y_2$  and summing over all  $Y_3$  containing  $y_3$  obtaining a factor

$$\sup_{y_3} \sum_{Y_3 \ni y_3} |K(Y_3)| |Y_3|^{d_3-1} \equiv Q(d_3).$$

The factor of  $|Y_3|$  is not really there since  $d_3 - 1 = 0$  but we include it to make our subsequent procedure more uniform. Next we sum over  $y_3$  in  $Y_2$  which has the effect

$$K(Y_2) \rightarrow |K(Y_2)| |Y_2|$$

(this is why we are including  $|Y|^{d-1}$  factors). Now we sum over  $Y_2$  containing a fixed point  $y_1$  in  $Y_1$  and obtain a factor  $Q(d_2)$ . In this way we can progressively do all the sums over  $Y$ 's, stripping off the branches.

One of several possible orders is  $Y_3, Y_2, Y_5, Y_4, Y_1$  which yields an upper bound

$$Q(d_3)Q(d_2)Q(d_5)Q(d_4) \sum_{Y_1 \subset A} |K(Y_1)| |Y_1|^{d_2}.$$

We can always arrange for  $Y_1$  to be the last polymer summed over. We can bound the sum over  $Y_1$  by summing first over all  $Y_1$  containing a fixed point  $y_1$  and then summing  $y_1$  over  $A$  obtaining a factor  $|A|$ .

Thus, the contribution of the graph at fig. 5 is less than

$$|A| Q(d_1 + 1) \prod_{i=2}^5 Q(d_i)$$

and a general tree on  $N$  vertices would be bounded by the same expression with 5 replaced by  $N$ . This bound leads directly to (3.10a) and theorem 3.4 after summing over  $d_1, \dots, \mathbf{d}, \bullet$ .

Theorem 3.5 is a simple variation on the same theme.

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