Time-dependent Subdiffusion via Continuous Time Random Walk Limits

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In physics literature, Continuous Time Random Walks (CTRWs) with infinite mean waiting times have been extensively used to model “subdiffusion”, i.e. any transport process whose variance grows at a sublinear rate $O(t^\beta)$, $\beta \in (0,1)$. The densities of the laws of such CTRW limit processes solve certain fractional diffusion equations which, in physics, are the main tool to describe the evolution of a subdiffusive system. For the general case of a CTRW X_t subject to an external force field $F(x,t)$ depending on *both* the space and the time variable, a governing fractional differential equation does not seem to exist.

The difficulty seems to lie in the non-locality in time of a fractional derivative. By keeping track of the spent waiting time V_t of a particle, a description that is Markovian and thus local in time is possible. For constant $F(t,x)$, we show that (X_t, V_t) converges to a limit process if the jump sizes and waiting times tend to 0. In the case where the waiting times and jumps are independent and where $F(x,t)$ is piecewise constant in time, we give a family of transition operators for this limit process. Finally, we assume a general force-field $F(x,t)$ and define the time-dependent subdiffusion process as the solution to a martingale problem.