Lecture 12

Proof continued of Prop. 11.7,

$\exists\ c(L)$ s.t. if

$$g \leq c(L)$$
$$\|K\|_h \leq c(L)$$

THEN

$$\lim_{L\to\infty} \frac{\|\tilde{K} - K_{main}\|_{\tilde{h}}}{\|K\|_h} = 0 \qquad (e: \text{prop } 11.7)$$

We wrote $\tilde{K} - K_{main} = I + II + III$. In the last lecture we proved that the contribution of III is zero. I will omit the argument for II unless I have time at the end.

Consider

$$I = \sum_{y\in B} \hat{L}^{-1}\left(e^{-V(B\setminus\{y\})} K_y\right)$$

The key here is

$$K(\phi_y) = \int_0^1 \frac{(1-t)^5}{5!} \left(\frac{d}{dt}\right)^6 K(t\phi_y)\, dt$$

$$= \int_0^1 \cdots\ K^{(6)}(t\phi_y)\, \phi_y^6\, dt$$

Note that

$$\hat{L}^{-1} e^{-V(B\setminus\{y\})} = e^{-(|B|-1)\, g\left(L^{-[\phi]}\phi\right)^4 + \cdots}$$

from $:\phi^4:$, $:\phi^2:$

Preliminary calculation

$$\tilde{h} = 2\tilde{g}^{-1/4}, \qquad \tilde{g} = L^{d-4[\phi]} g$$

$$(*) \quad L^{-[\phi]} \tilde{h} = 2 |B|^{-1/4} h$$

$$\left\| L^{-6[\phi]} \phi^6 \, e^{-(|B|-1) g \, L^{-4[\phi]} \phi^4} \right\|_{T_\phi, \tilde{h}}$$

$$\overset{*}{=} \left(L^{-[\phi]} \tilde{h} \right)^6 \left\| (\phi / \tilde{h}) \right\|_{\tilde{h}}^6 \left\| e^{-O(\tilde{g}) \phi^4} \right\|_{T_\phi, \tilde{h}}$$

$$\leq c \left(|B|^{-1/4} h \right)^6 \qquad (\text{c.f. Lemma 11.6})$$

so

$$\| I \|_{T_\phi, \tilde{h}} \leq c |B| \left(|B|^{-1/4} h \right)^6 \sup_t \| K^{(6)} \|_{L^{-[\phi]} \tilde{h}}$$

By Cauchy estimate from Lemma 9.4,

$$\leq c |B| \left(|B|^{-1/4} h \right)^6 \frac{1}{(h - L^{-[\phi]} \tilde{h})^6} \| K \|_h$$

$$\overset{(*)}{\leq} c |B| \left(|B|^{-1/4} h \right)^6 \left(\frac{1}{h - 2|B|^{-1/4} h} \right)^6 \| K \|_h$$

$$= O\left(|B|^{-1/2} \right) \| K \|_h = L^{-d/2} \| K \|_h$$

$|B| = L^d$ so prefactor $\to 0$ and so the contribution to (e:prop11.7) from I tends to zero

If we look at this proof again we find that term I dominates so in fact we have proved

<u>Proposition 11·7'</u> $\exists\ C(L)$ s.t

$g \le C(L)$,
$\|K\|_h \le C(L)$

$\Rightarrow$

$$\frac{\|\tilde{K} - K_{main}\|_{\tilde{h}}}{\|K\|_h} = O(L^{-d/2})$$

Remark 1

Proposition 11.7 required $K = O(\phi^6)$
$\tilde{K}$ will not obey this condition so we cannot use Proposition 11.7 for the next RG.

Define (V', K'), where

$$V' = g' :\phi^4: + a' :\phi^2: + b'$$

so that

$$e^{-V'} + K' = e^{-\tilde{V}} + \tilde{K}$$

$$K'(\phi_x) = O(\phi_x^6)$$

To see that a solution (V', K') exists, define V' by matching Taylor expansions in

$$e^{-V'} = e^{-\tilde{V}} + \tilde{K}$$

to order ϕ^4 and then let

$$K' = e^{-\tilde{V}} - e^{-V'} + \tilde{K}$$

Lemma 2

The solution (V', K') satisfies

(i) $\|V' - \tilde{V}\|_{T_0, h} \leq c \|\tilde{K}\|_{T_0, h}$

(ii) $\|K'\|_{\tilde{h}} \leq c \|\tilde{K}\|_{\tilde{h}}$

(iii) $\|K'\|_{T_0, h} \leq c \|\tilde{K}\|_{T_0, h}$

where $\tilde{h}$ was already defined and

$$h = L^{[\Phi]}.$$

Proof p569 \cite{bry2003BrIm03a}

Now we can prove that K_{main} controls K.

Corollary 3 For L large, $g \leq c(L)$,

If

z found by calculating $\| K_{main} \|_{\tilde{h}}$

$$c \| K_{main} \|_{\tilde{h}} \leq \tilde{g}^{z}$$

$$\| K \|_{h} \leq 2 \, g^{z}$$

and if $z(d - 4[\phi]) > -d/2$, THEN

$$\| K' \|_{h} \leq 2 \, \tilde{g}^{z}$$

Proof

$$\| K' \|_{\tilde{h}} \leq c \, \| \tilde{K} \|_{\tilde{h}} \qquad \text{(Lemma 2)}$$

$$\leq c \, \| \tilde{K} - K_{main} \|_{\tilde{h}} + c \| K_{main} \|_{\tilde{h}}$$

$$\leq O(L^{-d/2}) \, \| K \|_{h} + \tilde{g}^{z}$$

$$\leq O(L^{-d/2}) \, g^{z} + \tilde{g}^{z}$$

$$\leq O\left(L^{-d/2 + (d - 4[\phi])z}\right) \tilde{g} + \tilde{g}^{z}$$

$$= 2 \, \tilde{g}^{z} \qquad \text{for } L \text{ large}$$

Evolution of V

Lemma 2 (i) says that

$$g' = L^{d-4[\Phi]} g + O\left(h^{-4} \|\tilde{K}\|_{T_0, h}\right)$$

$$a' = L^{d-2[\Phi]} a + O\left(h^{-2} \|\tilde{K}\|_{T_0, h}\right)$$

(e: flow)

The next task is to prove that the corrections to linear terms are $o(g)$.

<u>Notation</u>

$h = g^{-1/4}$ as before

$\hbar = L^{[\Phi]}$

<u>Proposition 4</u>

Let $p > 0$. $\exists$ $c_p(L)$ s.t if

$g \leq c_p(L)$

$\|K\|_{T_0, \hbar} \leq c_p(L)$

THEN

$$\frac{\|\tilde{K} - K_{main}\|_{T_0, \hbar}}{\|K\|_{T_0, \hbar} \vee \left(h^{-p} \|K\|_h\right)} = O\left(L^{-d/2}\right)$$

By choosing $p = 12$, $h^{-p} = g^3$ which is so small that $h^{-p} \cdot \|K\|_h$ will drop out and as in Remark 3

Domain

Let $\delta > 0$, $L \geq L_0(\delta)$

$$g \leq c(L)$$
$$|a| \leq g$$
$$\|K\|_{T_0, h} \leq g^{2-\delta}$$
$$\|K\|_h \leq g^{\frac{1}{2}-\delta}$$

based on calculating $\| K_{main} \|$

Then

$$g' = L^{d-4[\phi]} g + \mathcal{E}_g \qquad \mathcal{E}_g \leq g^{2-\delta}$$

$$a' = L^{d-2[\phi]} a + \mathcal{E}_a \qquad \mathcal{E}_a \leq g^{2-\delta}$$

$$b' = L^{d} b + \mathcal{E}_b \qquad \mathcal{E}_b \leq g^{2-\delta}$$

and K' obeys g' version of above

Furthermore this is hyperbolic so there exists a critical choice of a_0, b_0 s.t. action of RG sends $g, a, b \to 0$.

Notation

$$\mathbb{E}^{(p-1)} F = \sum_{n=0}^{p-1} \frac{1}{n!} \left(\frac{\Delta_C}{2}\right)^n F$$

Lemma 5

$$\| \mathbb{E}_1 F - \mathbb{E}^{(p-1)} F \|_{T_\phi, h_1}$$

$$\leq \frac{(2p)!}{2^p p!} \left(\frac{C(0,0)}{h_2^2}\right)^p \| F \|_{h_1 + h_2}$$

Proof

$$\mathbb{E}_1 F = \mathbb{E}^{(p-1)} F + \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \left(\frac{d}{dt}\right)^p \mathbb{E}_t F$$

where $\mathbb{E}_t$ has covariance tC in place of C

$$= \mathbb{E}^{(p-1)} F + \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} \mathbb{E}_t \left(\frac{\Delta_C}{2}\right)^p F$$

so

$$\| \mathbb{E}_1 F - \mathbb{E}^{(p-1)} F \|_{T_\phi \, h_1}$$

$$\leq \frac{1}{p!} \sup_t \| \mathbb{E}_t \left(\frac{\Delta_C}{2}\right)^p F \|_{T_\phi, h_1}$$

$$\leq \frac{1}{p!} \left\| \left(\frac{\Delta_C}{2}\right)^p F \right\|_{h_1}$$

$$\leq \frac{1}{p!} \frac{(2p)!}{2^p h_2^{2p}} \left(C(0,0)\right)^p \|F\|_{h_1+h_2} \qquad (\text{Lemma } 11.4)$$

because $C(x,y) \leq C(0,0)$ by Cauchy-Schwarz and positive-definiteness

By taking $p=1$ we obtain a bound on $\mathbb{E}_C F - F$ by $O(h^{-2}) \|F\|_{2h}$ which is what is needed to bound term II in the proof of Proposition 11.7.

Part of Proof of Proposition 4

(1) If $F = F(\phi_x)$

and $F^{(n)}(0) = 0$ for $n = 0, 1, \cdots, p-1$

then

$$\|F\|_{T_0, \alpha h} = \sum_{n \geq p} \frac{1}{n!} (\alpha h)^n \, |F^{(n)}(0)|$$

$$\leq \alpha^p \, \|F\|_{T_0 \, h}$$

(2)

$$\tilde{K} - K_{main} = \sum_{y \in B} \hat{L}^{-1} E_1 \left(e^{-V(B \setminus \{y\})} K_y \right)$$

$$+ \sum_{\substack{Y \subset B \\ |Y| \geq 2}} \hat{L}^{-1} E_1 \left(e^{-V(B \setminus Y)} K^Y \right)$$

Let $F = e^{-V(B \setminus \{y\})} K_y$. Then $\hat{L}^{-1} F = O(\phi^6)$

$$\| \hat{L}^{-1} E_1 F \|_{T_0, h}$$

$$\leq \| E_1 F \|_{T_0 \, L^{-[\phi]} h}$$

$$\overset{\text{Lemma 5}}{\leq} \sum^{p-1} \frac{1}{n!} \left\| \left(\frac{\Delta c}{2} \right)^n F \right\|_{T_0, L^{-[\phi]} h} + O\left(\frac{h}{h} \right)^p \|F\|_h$$

$$\leq C(p) \, \|F\|_{T_0 \, 2L^{-[\phi]} h} + O\left(\frac{h}{h} \right)^p \|F\|_h$$

$$\overset{(1)}{\leq} O\left(L^{-6[\phi]}\right) \|F\|_{T_0, h} + O\left(\frac{h}{h} \right)^7 \|F\|_h$$

$$\leq O(L^{-6[\phi]}) \; \|K\|_{T_0, \hbar} + O\left(\frac{\hbar}{h}\right)^p \|K\|_{h}$$

By ② the contribution to $\|\tilde{K} - K_{\text{main}}\|_{T_0, \hbar}$ is, using $|B|$ to count terms in $\sum_{y \in B}$,

$$|B| \; O(L^{-6[\phi]}) \; \|K\|_{T_0, \hbar} + O\left(\frac{\hbar}{h}\right)^p \|K\|_h$$

$$\leq O\left(L^{d-6[\phi]}\right) \; \|K\|_{T_0, \hbar} \; \vee \left(h^{-p+1} \|K\|_h\right)$$

where we used $h \geq \hbar^p$, which is true by $h = g^{-1/4}$ and $g \leq c(L)$, where we can choose $c(L)$ so that $\hbar^p / h \leq 1$. Since this holds for all p, we can change p back to $p-1$.

Since

$$L^{d-6[\phi]} \to 0, \quad L \to \infty$$

this contribution to $\|\tilde{K} - K_{\text{main}}\| \,/\, \|K\|_{T_0, \hbar} \vee \left(h^{-p} \|K\|_h\right)$ $\to 0$ as $L \to \infty$

$\boxtimes$