Lecture 14

Example 1

What does $\tilde{K}$ reduce to if the metric is hierarchical?

Lemma B.6 still applies because it made no assumption about the metric.

$$\tilde{K}(U) = \prod_{B \in \mathcal{B}_{j+1}(U)} \tilde{K}(U)$$

because blocks are connected components of U.

$$\tilde{K}(B) = \sum_{X \in \overline{\mathcal{P}}_j(B)} \tilde{I}^{B \setminus X} \, \mathbb{E}_{j+1} (SI \circ K)(X)$$

$$= \sum_{X_K, X_{SI} \in \mathcal{P}_j(B)} 1_{X_K \cup X_{SI} = B}$$

$$\tilde{I}^{B \setminus (X_K \cup X_{SI})} \; \mathbb{E}_{j+1} \; SI^{X_{SI}} \; K^{X_K}$$

Look at the part that does not contain K

Work out the part that does not contain any K

$$= \sum_{X_{\delta I}} 1_{\bar{X}_{\delta I} = B} \; \tilde{I}^{B \setminus X_{\delta I}} \; \mathbb{E}_{j+1} \; \delta I^{X_{\delta I}}$$

$$= \mathbb{E}_{j+1} \left(\tilde{I} + \delta I \right)^{B} - \tilde{I}^{B}$$

reverse the binomial exp.

$$= \mathbb{E}_{j+1} \; I^{B} - \tilde{I}^{B}$$

$$= \mathbb{E}_{j+1} \; e^{-V(B)} - e^{-\mathbb{E}_{j+1} V(B)} = \underline{\underline{K_{main}}}$$

End of Exampl

<u>Remark 2</u>

$$\delta I^{X} = \prod_{b \in \mathcal{B}_j(X)} \delta I(b)$$

$$\delta I(b) = I(b) - \tilde{I}(b)$$

(no $\hat{L}^{-1}$ to collapse b to a point)

Defn 3

$$\mathcal{S}_j = \{ X \in \mathcal{P}_j : X \text{ connected } \; |X|_j \le 2^d \}$$

$$|X|_j = |\mathcal{B}_j(X)|$$

For $B \in \mathcal{B}_j$,

$$B^* = \bigcup \{ Y \in \mathcal{S} : Y \supset B \}$$

For $X \in \mathcal{P}_j$

$$X^* = \bigcup \{ B^* : B \in \mathcal{B}_j(X) \}$$

X^* is called the small set neighbourhood of X.

For $U \in \mathcal{P}_{j+1}$, we say

$$X \in \bar{\mathcal{S}}_j(U)$$

if

$$\bar{X} = U \text{ and } X \in \mathcal{S}_j$$

The following geometric lemmas hold for $L \geq L_0(d)$.

Lemma 4 $\exists\, c > 1$ s.t.

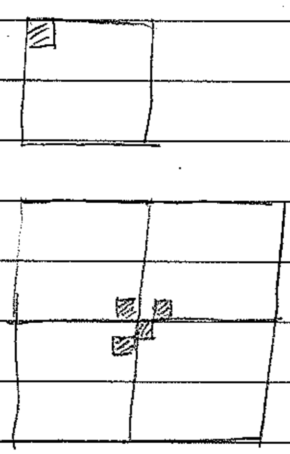
If $X \notin S$ and X is connected

$$|X|_j \geq c\,|\bar{X}|_{j+1}$$

Lemma 5 $\exists\, c \geq 1$ s.t.

$$|X|_j \geq c\,|\bar{X}|_{j+1} - c\,2^{d+1}\,n(X)$$

where $n(X) = \#$ of components of X, $X \in \mathcal{P}_j$.



$$|\bar{X}|_{j+1} = |X|_j$$

Let

$$\tilde{K}_{main}(U)$$

$$= \sum_{X \in \bar{\mathcal{P}}_j(U)} \tilde{I}^{U \setminus X} \; \mathbb{E}_{j+1}(\delta I)^X$$

(the $K = 0$ contribution)

Let

$$* = \left\{ (X_K, X_{\delta I}) \in \mathcal{P}_j^2(U) \text{ s.t. } \overline{X_K \cup X_{\delta I}} = U, \quad \overset{n(X_K)}{\cancel{n_K}} \geq 2 \quad X_K \cap X_{\delta I} = \phi \right\}$$

$\overset{n(X_K)}{\cancel{n_K}}$ = # comp. of X_K

Let

$$R_*(U) = \sum_* \tilde{I}^{U \setminus X_K \cup X_{\delta I}} \; \mathbb{E}_{j+1}\left(K(X_K) \; \delta I^{X_{\delta I}} \right)$$

Norm Suppose we have norm(s) s.t

(1) $\| F(X)\, G(Y) \| \leq \| F(X) \| \, \| G(Y) \|$

for X, Y disjoint

(2) $\| E_{j+1} (\delta I)^{X} F(Y) \|$

$$\leq \alpha^{|X|_j + |Y|_j} \epsilon_{SI}^{|X|_j} \| F(Y) \|$$

(3) $\| \tilde{I}^{X} \| \leq \alpha^{|X|_j}$

We let ϵ_K be best constant s.t.

$$\| K(X) \| \leq \epsilon_K^{n(X)} A^{-|X|_j}$$

Lemma 6 $\exists\ \delta > 0$ and $\exists\ c(A)$ s.t

$$\lim_{\substack{A \to \infty \\ \epsilon_K \leq c(A) \\ \epsilon_{SI} \leq c(A) \\ \cancel{L \text{ fixed}}}} \frac{1}{\epsilon_K} \| R_*(U) \| \, A^{-(1+\delta)|U|_{j+1}} = 0$$

In other words, for A suff. large, $\epsilon_K \leq c(A)$, $\epsilon_{SI} \leq c(A)$,

$$\| R_*(U) \| \leq 10^{-10} \epsilon_K A^{-(1+\delta)|U|_{j+1}}$$

Proof

Preliminary calculation: Lemma 5 implies

$$A^{-|X_K|_j - |X_{\delta I}|_j}$$

$$\leq A^{-c|\overline{X_K \cup X_{\delta I}}|_{j+1}} \; A^{c2^{d+1}\left(n(X_K) + n(X_{\delta I})\right)}$$

$$\leq A^{-c|U|_{j+1}} \; A^{c2^{d+1}\left(n(X_K) + |X_{\delta I}|\right)}$$

because $n(X_{\delta I}) \leq |X_{\delta I}|$

Then (some corrections to what I wrote in class because of factors of A associated with $\epsilon_{\delta I}$)

$\|R_*(U)\|$

$$\leq \alpha^{|U|_j} \sum_* \epsilon_K^{n_K} A^{-|X_K|_j} (\epsilon_{\delta I} A)^{|X_{\delta I}|_j} A^{-|X_{\delta I}|}$$

$$\overset{\text{Lemma } 5}{=} \alpha^{|U|_j} \sum_* A^{-c|\overline{X_K \cup X_{\delta I}}|} \left(A^{c2^{d+1}} \epsilon_K\right)^{n_K} \underbrace{\left(A^{c2^{d+1}+1} \epsilon_{\delta I}\right)^{n_{\delta I}}}_{\leq 1 \text{ by choice of } C(A).}$$

$$\leq \alpha^{|U|_j} A^{-c|U|_{j+1}} \left(A^{c2^{d+1}} \epsilon_K\right)^2 \sum_* 1$$

$$\leq \alpha^{|U|_j} A^{-c|U|_{j+1}} \left(A^{c2^{d+2}} \epsilon_K\right) \epsilon_K \; 3^{|U_j|}$$

$$= \underbrace{(3\alpha)^{|U|_j} A^{-(c-1-\delta)|U|_{j+1}}}_{\substack{\le \\ \left((3\alpha)^{L^d} A^{-(c-1-\delta)}\right)^{|U|_{j+1}} \\ \longrightarrow 0 \text{ as } A \to \infty \\ L \text{ fixed.}}} \underbrace{\left(A^{c2^{d+2}} \epsilon_K\right)}_{\substack{\le 1 \\ \text{by } \epsilon_K \le c(A)}} \left(\epsilon_K A^{-(1-\delta)|U|_{j+1}}\right)$$

Lemma 6'

The same as Lemma 6, in (*) replace $n_K \ge 2$ by

$$n_K \ge 1, \quad \text{if } n_K = 1 \text{ then } X_K \notin S.$$

Problem

1) Use Lemma 4 to deduce Lemma 6'