Lecture 15

In the hierarchical model we used a representation for the interaction

$$e^{-V_j} + K_j \qquad\qquad K_j = O(\phi^6)$$

The RG action $(V_j, K_j) \longrightarrow (V_{j+1}, K_{j+1})$ was constructed in two stages

$$(V_j, K_j) \xrightarrow{①} (\tilde{V}, \tilde{K}) \xrightarrow{②} (V_{j+1}, K_{j+1})$$

Today we discuss the Euclidean analogue of ②.

Define

$$\tilde{\mathcal{L}}(U) = \sum_{X \in \overline{S}_j(U)}' \tilde{I}^{U \setminus X} \mathbb{E}_{j+1} K(X) \qquad (e:Ltilde)$$

Putting this definition together with our work in the last lecture we find that the action of RG in (V, K) coordinates,

$$\mathbb{E}_{j+1}(I_j \circ K_j)(\Lambda)$$

$$= \tilde{I} \circ \tilde{K}(\Lambda)$$

is given by

$$I_j = e^{-V_j}, \quad \tilde{I} = e^{-\tilde{V}}, \quad \tilde{V} = \mathbb{E}_{j+1} V$$

$$\tilde{K} = \tilde{K}_{main} + \tilde{\mathcal{L}} + R_*$$

where

$$\tilde{K}_{main}(U) = \sum_{X \in \overline{\mathcal{P}}_j(U)} \tilde{I}^{U \setminus X} \mathbb{E}_{j+1} \delta I^X$$

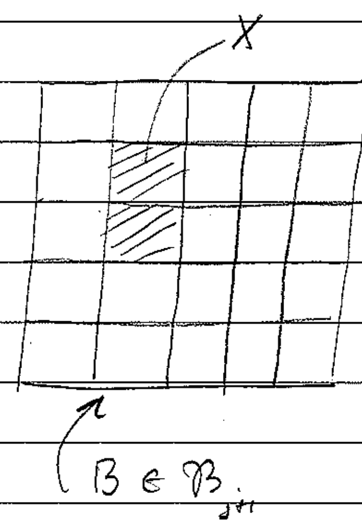
is a function only of V and R_* is a neglible proportion of $\tilde{K}$ (Prop. 14-6).

Following the hierarchical "yellow brick road" we want to find a domain for (V, K) where RG is norm bounded and the main step to achieve this will be to prove that $\tilde{\mathcal{L}}$ is contractive.

<u>Remark 1</u> Since we are not rescaling the norms will contain the rescaling and so will vary with j. We will get to this later.

In (e: Ltilde) the $\sum\limits_{X \in \bar{S}(U)}$ has $O(L^d)$ terms

This sets the stage for an $O(L^d)$ expansion in the norm.



The same issue in the hierarchical case led us to impose

$$K = O(\phi^6)$$

because this gives a compensating $L^{-6[\phi]}$ in term I, proof of Prop. 11-7.

X has $O(L^d)$ possible position in B

Then, in order to have the $K = O(\phi^6)$ at the next scale we solved

$$e^{-\tilde{V}} + \tilde{K} = e^{-V_{j+1}} + K_{j+1}$$

Example 2

The analogous procedure for the Euclidean case is to adjust $\tilde{V}$ to V' in such a way that

$$e^{-\tilde{V}} \circ \tilde{K} = e^{-V'} \circ K'$$

$$K'(B) = O(\phi^6), \quad B \in \mathcal{B}_{j+1}$$

In more detail:

$$e^{-\tilde{V}} \circ \tilde{K} = (\underbrace{e^{-V'}}_{I'} + \underbrace{e^{-\tilde{V}} - e^{-V'}}_{\delta\tilde{I}}) \circ \tilde{K}$$

$$= (I' \circ \delta\tilde{I}) \circ \tilde{K}$$ c.f. Lecture 13

$$= I' \circ \underbrace{(\delta\tilde{I} \circ \tilde{K})}_{:= K'}$$

Thus

$$K'(B) = (\delta\tilde{I} \circ \tilde{K})(B)$$

$$= \delta\tilde{I}(B) + \tilde{K}(B)$$

$$= e^{-\tilde{V}(B)} - e^{-V'(B)} + \tilde{K}(B)$$

and we can adjust $V'(B)$ so that this is $O(\phi^6)$, for ϕ = constant on B.

This, however, will not solve the problem of transferring from scale j to j+1 all the conditions

$$K_j(X) = O(\Phi^6), \qquad X \in S_j$$

How to do this is the key problem to be surmounted in the Euclidean case. The solution I am about to describe is contained in my on-going work with Gordon Slade, but it evolved from a more primitive idea in \cite{BrydgesYau1990}.

Let

$$J = \{ J(X) : X \in \mathcal{P}_{j+1} \}$$

$$J(X) = 0 \quad \text{if } X \notin \mathcal{S}_{j+1}$$

$$\sum_{X \supset B} \frac{1}{|X|_{j+1}} I^{-X} J(X) = 0 \qquad \text{(e: Jcondition)}$$

Let ϵ

$$\| J(X) \| \leq \epsilon A^{-|X|_{j+1}}$$

$$\| \tilde{K}(X) - J(X) \| \leq \epsilon A^{-(1+\delta)|X|_{j+1}}$$

Proposition 3

$\exists\ c(A)$ and $\exists\ K'$ s.t

$$\lim_{\substack{A \to \infty \\ \epsilon \leq c(L)}} \epsilon^{-1} A^{(1+\delta/2)|X|_{j+1}} \| K'(X) - (\tilde{K}(X) - J(X)) \| = 0$$

and

$$\tilde{I} \circ \tilde{K} = \tilde{I} \circ K'$$

and K' factors.

This solves the problem of arranging for

$$K'(X) = O(\phi^6)$$

$$\forall\ X \in S_{j+1} \setminus \mathcal{B}_{j+1}$$

because we can choose $\{J(X) : X \in S_{j+1} \setminus \mathcal{B}_{j+1}\}$ so that

$$\tilde{K}(X) - J(X) = O(\phi^6), \quad X \in S_{j+1} \setminus \mathcal{B}_{j+1}$$

(on ϕ = constant).

The relation $\sum \frac{1}{|X|_{j+1}} \tilde{I}^{-X} J(X) = 0$ then determines $J(B)$. Therefore we will not have the desired

$$K'(B) = O(\phi^6)$$

but this is the problem we know how to solve by adjusting $\tilde{I}(B)$ as in Example 2.

Proof

Construction of K'

Given $W \in \mathcal{P}_{j+1}$ let $\mathcal{Y}(W)$ be the set of triples

$$\left(X, \vec{U}, U_M\right)$$

where

1) $X \in \mathcal{P}_{j+1}(W)$

2) $\vec{U} = \{ U(B) : B \in \mathcal{B}_{j+1}(X),\ U \supset B,\ U \in \mathcal{S}_{j+1} \}$

3) $U_M \in \mathcal{P}_{j+1}(W)$

4) strict disjointness:

5) $X^* \cup U_M = W$

6) triples with $|X| = 1$, $U_M = \phi$ are omitted.

Do not read these now. They describe constraints arising in sums below.

Since $\tilde{K} = J + M$, where $M = \tilde{K} - I$,

$(\tilde{K} \circ \tilde{I})(\Lambda)$

$$= \sum_{\tilde{U} \in \mathcal{P}(\Lambda)} \left(\prod_{U \text{ c.c. } \tilde{U}} (J(U) + M(U)) \right) \tilde{I}^{\Lambda \setminus \tilde{U}}$$

$$= \sum_{U_J, U_M} \left(\prod_{U \text{ c.c. } U_J} J(U) \right) \left(\prod_{V \text{ c.c. } U_M} M(V) \right) \tilde{I}^{\Lambda \setminus (U_J \cup U_M)}$$

Insert, for U c.c. U_J,

$$J(U) = \sum_{B \in \mathcal{B}(U)} \frac{1}{|\mathcal{B}(U)|} J(U)$$

This creates a sum over pairs $\{(B, U(B)) : U(B) \in \text{c.c. } U_J\}$

Let X be the union of these B's.

Let $W = X^* \cup U_M$. Then

$(\tilde{K} \circ \tilde{I})(\Lambda)$

$$= \sum_{W \in \mathcal{P}(\Lambda)} \left(\sum_{(X, \vec{U}, U_M) \in \mathcal{Y}(W)} \left(\prod_{B \in \mathcal{B}(X)} \frac{1}{|U(B)|_{J+1}} J(U(B)) \right) \left(\prod_{V \text{ c.c. } U_M} M(V) \right) \tilde{I}^{W \setminus U' \cup U_M} \right) \tilde{I}^{\Lambda \setminus W}$$

with

$$U' = \bigcup_{B \in \mathcal{B}(X)} U(B).$$

Let $K'(U)$ be the factor in the huge parenthesis so that the equation reads

$$\tilde{K} \circ \tilde{I} = K' \circ \tilde{I}$$

as desired.

In $K'(W)$
consider the terms where $U_M = \emptyset$, $X = B$. They are

$$\sum_{U(B) \supset B} \frac{1}{|U(B)|_{j+1}} J(U(B)) = 0$$

by (e: Jcondition). Therefore condition (6) holds.

The bound on

$$\| K'(W) - (\tilde{K}(W) - J(W)) \|$$

Looking at the formula for K' we see that the contribution to K' when $X = \emptyset$ cancels with $\tilde{K} - J$ because $M = \tilde{K} - J$ for U_M with one component so $K'(W) - (\tilde{K}(W) - I)$ is second order in ϵ. These higher order terms are bounded using the same ideas as were used in the proof of Prop. 14.6. See \cite{bry2009Brydges} for the proof of similar results.

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