The Gaussian Field Cont.

We have seen that Gaussian fields can be used to represent particle systems in a way that reveals the role of the Kac limit and mean field theory. Many other models also have Gaussian field representations. Today we will see, as further examples, representations of self-avoiding "molecules". This also gives me a chance to briefly explain "Fermions" but my explanation is a device to get you used to the idea of "anticommuting variables" as a useful combinatorial tool by equating them with differential forms. The book by Berezin describes the standard setup which does not insist on any identification with differential forms.

Oriented Edges

Until now $A_{xy} = A_{yx}$
so $C_{xy} = C_{yx}$ so graphs have unoriented edges. To obtain oriented edges:

Let $A = (A_{xy})$ be a not necessarily symmetric matrix s.t

$$\mathrm{Re}\,(\phi, A\bar{\phi}) > 0 \qquad \phi \neq 0, \quad \phi \in \mathbb{C}^{\Lambda}$$

Complex valued ϕ can "see" the antisymmetric part of A.

Define

$$d\mu_C(\phi) = \frac{1}{N} e^{-(\phi, A\bar{\phi})} d^{2\Lambda}\phi$$

$$C = A^{-1}, \qquad N = \pi^{|\Lambda|} \det^{-1} A$$

$$d^{2\Lambda}\phi = \prod_{x \in \Lambda} du_x \, dv_x$$

$$\phi_x = u_x + i v_x$$

Remark
If $A_{xy} = A_{yx}$ real,

$$(\phi, A\bar{\phi}) = (u, Au) + (v, Av)$$

$$N \propto \det^{-1/2} A \, \det^{-1/2} A = \det^{-1} A$$

Lemma 1 $F \in C'$, both sides integrable,

$$\int d\mu_C \, \bar{\Phi}_a F = \int d\mu_C \sum_{x \in \Lambda} C_{ax} \frac{\partial F}{\partial \phi_x}$$

Notation $\frac{\partial}{\partial \phi} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v}\right)$

so that

$\frac{\partial}{\partial \phi} \phi = 1$

$\frac{\partial}{\partial \phi} \bar{\phi} = 0$

Proof

$$\int \bar{\Phi}_a e^{-(\phi A \bar{\Phi})} F \, d^{2\Lambda}\phi$$

$$= \int ((CA\bar{\phi})_a \, e^{-(\phi, A\bar{\phi})} F \, d^{2\Lambda}\phi$$

$$= \sum_x C_{ax} \int \left(-\frac{\partial}{\partial \phi_x} e^{-(\phi, A\bar{\Phi})}\right) F \, d^{2\Lambda}\phi$$

$$= \sum_x C_{ax} \int e^{-(\phi, A\bar{\phi})} \frac{\partial F}{\partial \phi_x} d^{2\Lambda}\phi$$

∎

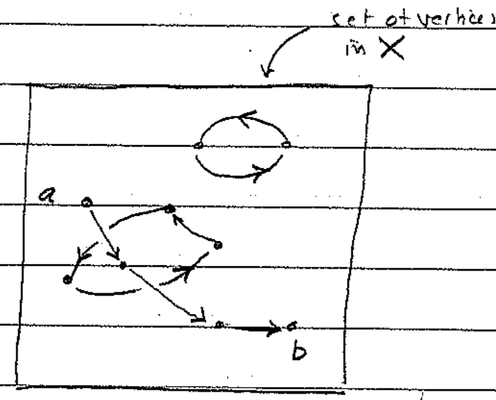
As in Lecture 5 we have a Wick's theorem with $\exp\left(\sum C_{xy} \frac{\partial}{\partial \bar{\Phi}_x} \frac{\partial}{\partial \phi_y}\right)$, but this Lemma is another form of Wick's theorem

Example 2 $\int d\mu_C \, \bar{\Phi}_a \phi_b = C_{ab}$

Example 3

$$I_X = \int d\mu_C \prod_{\substack{x \in X \\ x \neq a,b}} \left(1 + :\phi_x \bar{\Phi}_x:\right) \quad \bar{\Phi}_a \phi_b$$

$$= \sum_{G \in \mathcal{G}_{ab}} \prod_{(x,y) \in G} C_{xy}$$



$G \in \mathcal{G}_{ab}$

Every $x \neq a,b \in \Lambda$ has no edges

or one incoming, one outgoing

$x = a$ one outgoing,

$x = b$ one incoming

i.e. $G \in \mathcal{G}$ iff $\exists$ self-avoiding walk from a to b

and an arbitrary number of cycles, all disjoint.

Proof Induction on X

By Lemma 1,
$$I_X = \sum_{x_1} C_{ax_1} \int d\mu_C \prod_{\substack{x \in X \setminus \{a\} \\ x \neq x_1, b}} \left(1 + :\phi_x \bar{\Phi}_x:\right) \bar{\Phi}_{x_1} \phi_b$$
$$+ \int d\mu_C \prod_{x \in X} \left(1 + :\phi_x \bar{\Phi}_x:\right)$$

Apply inductive hypothesis to first term.

Second term:

$$\int d\mu_C \prod_{x \in X} \left(1 + :\phi_x \bar{\Phi}_x:\right) =$$

$$= \int d\mu_C \prod_{x \in X \setminus \{x_1\}} \left(1 + :\phi_x \bar{\Phi}_x\right)^2 + \int d\mu_C \prod_{x \in X \setminus \{x_1\}} \left(1 + :\phi_x \bar{\Phi}_x:\right) \phi_{x_1} \bar{\Phi}_{x_1}$$

Apply induction.

Differential Forms = Fermions

The symbols

$$\left(du_{x\in\Lambda},\ dv_{x\in\Lambda}\right)$$

generate a finite dimensional algebra Ω^* over the ring of complex valued functions of $\phi_x = u_x + i v_x$, $x \in \Lambda$ via the wedge product

$$du_x \wedge du_y = - du_y \wedge du_x$$
$$du_x \wedge dv_y = - dv_y \wedge du_x$$
$$dv_x \wedge dv_y = - dv_y \wedge dv_x$$

Because $\wedge$ looks like Λ I omit $\wedge$. The degree of a form is the degree as a polynomial in du, dv.

Ω^* is called the algebra of differential forms.

Example 4 Define

$$d\phi_x = du_x + i dv_x, \qquad d\bar\phi_x = du_x - i dv_x$$

$$d\bar\phi_x \, d\phi_x = \left(du_x - i dv_x\right)\left(du_x + i dv_x\right)$$

$$= 2i \, du_x \, dv_x$$

Defn 5 The volume form on $\mathbb{C}^\Lambda = \mathbb{R}^{2\Lambda}$ is

$$\prod_{x\in\Lambda} (du_x \, dv_x) = (2i)^{-|\Lambda|} \prod_{x\in\Lambda} (d\bar{\phi}_x \, d\phi_x)$$

This is a top degree $(= 2|\Lambda|)$ form. The particular way we have written it removes a sign ambiguity which would result if we did not carefully specify the order in which the du, dv must be written.

Defn 6 For $F \in \Omega^*$, let $f(u,v) \prod_{x\in\Lambda} du_x \, dv_x$ be the top degree part of ω. Define

$$\int F = \int_{\mathbb{R}^{2\Lambda}} f(u,v) \, d^{2\Lambda}\phi$$

$$\left(\text{"} \prod_{x\in\Lambda} du_x \, dv_x = d^{2\Lambda}\phi \text{"} \right)$$

$d^{2\Lambda}\phi$ was defined before forms were introduced. It is Lebesgue measure

Notice that $\int \omega = 0$ if ω has zero top degree part.

Example 7 Let $N = |\Lambda|$

$$\left(\sum_{x,y\in\Lambda} A_{xy} \, d\bar{\phi}_x \, d\phi_y \right)^N$$

$$= \sum_{x_1,y_1} \sum_{x_2,y_2} \cdots \sum_{x_N,y_N} A_{x_1 y_1} \cdots A_{x_N y_N}$$

$$d\bar{\phi}_{x_1} \, d\phi_{y_1} \cdots d\bar{\phi}_{x_N} \, d\phi_{y_N}$$

$$= N! \, \det A \prod_{x\in\Lambda} d\bar{\phi}_x \, d\phi_x$$

Example 8 Let

$$S = (\phi, A\bar{\phi}) + \frac{1}{2\pi i} \sum_{x,y\in\Lambda} A_{xy}\, d\phi_x\, d\bar{\phi}_y$$

Define $e^{-S} \in \Omega^*$ by power series in the form part.

$$\int e^{-S} = \int e^{-(\phi, A\phi)} \sum \frac{1}{n!}\left(\frac{1}{2\pi i} \sum (-A_{xy})\, d\phi_x\, d\bar{\phi}_y\right)^n$$

$$\stackrel{\text{Ex 7}}{=} \det(A^t)\, \pi^{-N} \int e^{-(\phi, A\phi)}\, d^{2\Lambda}\phi$$

$$= 1$$

$$-A_{xy}\, d\phi_x\, d\bar{\phi}_y = A_{xy}\, d\bar{\phi}_y\, d\phi_x$$

This is self-normalisation!

Define $\tau_x \in \Omega^*$ by

$$\tau_x = \phi_x \bar{\phi}_x + \frac{1}{2\pi i}\, d\phi_x\, d\bar{\phi}_x$$

I claim that for all $X \subset \Lambda$

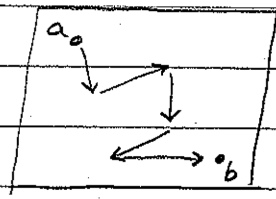
$$\int e^{-S} \prod_{x\in X} (1 + \tau_x) = 1 \qquad (*)$$

Believing this for now,

Example 9 SAW

$$\int e^{-S} \prod_{\substack{x\in\Lambda \\ x\neq a,b}} (1+\tau_x) \quad \bar{\phi}_a \; \phi_b$$

$$= \sum$$



SAW_{ab} = {all self-avoiding walks from a to b}.

$$= \sum_{\omega\in SAW_{ab}} \prod_{(x,y)\in\omega} C_{xy}$$

Sketch of proof $\int e^{-S} \prod (1+\tau_x) \; \bar{\phi}_a \phi_b$ is a sum of standard integrals in each of which we can apply Lemma 1

$$\bar{\phi}_a \rightarrow \sum_{x_1} C_{a x_1} \frac{\partial}{\partial \phi_{x_1}}$$

$$\frac{\partial}{\partial \phi_{x_1}} (1+\tau_z) = \bar{\phi}_{x_1} \quad \text{if } z=x_1$$

$$= 0 \quad \text{else}$$

By induction, get $\sum$ over SAW ω

$$\int e^{-S} \prod_{\substack{x\in\Lambda \\ x\notin\omega}} (1+\tau_x) = 1$$

by our claim.

Supersymmetry

Define

$$i_X : \Omega^* \longrightarrow \Omega^*$$

by

① i_X is an antiderivation

② i_X (zero form) $= 0$

③ $i_X \, d\phi_x = -2\pi i \, \phi_x$
$\quad\; d\bar{\Phi}_x = 2\pi i \, \bar{\Phi}_x$

i_X lowers degree. Recall the exterior derivative d is also an anti-derivation

Let

$$Q = d + i_X$$

Q is called the supersymmetry generator. If $F \in \Omega^*$ and $QF = 0$ we say F is supersymmetric.

Example 10 τ_x is supersymmetric

$$Q\,\tau_x = d\phi_x\,\bar\phi_x + \phi_x\,d\bar\phi_x$$

$$+ \frac{1}{2\pi i}\Big((-2\pi i\,\phi_x)\,d\bar\phi_x - d\phi_x\,(2\pi i\,\bar\phi_x) \Big)$$

$$= 0$$

Exercise

$$\tau_x = Q\left(\frac{\phi_x\,d\bar\phi_x}{2\pi i}\right)$$

Lemma 11 (Localisation)

Let $F \in \Omega^*$ be an even form (only even degree monomials) with smooth coefficients which together with derivatives decay integrably. If $QF = 0$ then

$$\int F = F(\text{set } \phi,\ d\phi,\ d\bar\phi = 0)$$

Note that this proves our claim (*)

Proof

$$\sum_{x\in\Lambda} \tau_x = Q\omega, \qquad \omega = \sum \frac{\phi_x \, d\bar\phi_x}{2\pi i}$$

$$\frac{d}{dt}\int F e^{-t\sum \tau_x}$$

$$= -\int F\,(Q\omega)\, e^{-t\sum \tau_x}$$

$$= -\int Q\left(F\omega\, e^{-t\sum\tau_x}\right)$$

antiderivation

$QF=0$, $Q\tau=0$

$$= -\underbrace{\int d(\cdots)}_{\text{Stokes Thm}} - \underbrace{\int i_X(\cdots)}_{\text{wrong degree}}$$

$$= 0 + 0$$

Therefore

$$\int F = \lim_{t\to\infty}\int F e^{-t\sum \tau_x}$$

$$= F(\phi=0,\ d\phi=0,\ d\bar\phi=0)$$

The last step is a homework problem.

Remark (Origin of term supersymmetry)

$$Q^2 = (d + i_X)^2$$

$$= d^2 + d \circ i_X + i_X \circ d + i_X^2$$

$d^2 = 0$ and i_X is also nilpotent, $i_X^2 = 0$

$$= d \circ i_X + i_X \circ d$$

$$= \mathcal{L}_X$$

where $\mathcal{L}_X$ is Lie derivative w.r.t vector field that generates $U(1)$ action

$$\phi \rightarrow \phi e^{-2\pi i \theta}$$

$Q^2 = \mathcal{L}_X$ says that Q is the square root of the $U(1)$ generator.

Problems

(1) Justify the last step in the proof of Lemma 11

(2) Let A be symmetric.

Define $B_x = \sum_y A_{xy}$

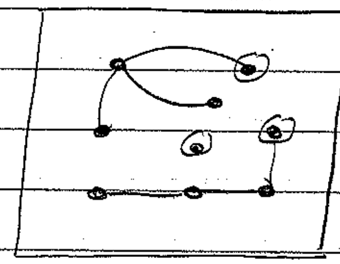
For $R \subset \Lambda$ let $B^R = \prod_{x \in R} B_x$

For a graph F let $(-A)^F = \prod_{xy \in Edges(F)} (-A_{xy})$

The matrix tree theorem says

$$\det A = \sum_{(F,R)} (-A)^F \, B^R$$

where F is summed over all graphs on Λ which have no cycles and for each F, R is summed over all ways to choose one root in each connected subgraph of F.



A possible (F,R). The circled vertices are the set R.

Prove the matrix tree theorem by starting with $\int e^{-S} = 1$.

Write

$$\sum_{x,y} A_{xy} \, d\phi_x \, d\bar{\phi}_y$$

$$= -\frac{1}{2} \sum_{x,y} A_{xy} (d\phi_x - d\phi_y)(d\bar{\phi}_x - d\bar{\phi}_y) + \sum B_x \, d\phi_x \, d\bar{\phi}_x$$

Write $\phi_{xy} = \phi_x - \phi_y$, $d\phi_{xy} = d\phi_x - d\phi_y$ and expand $e^{\sum A \, d\phi \, d\bar{\phi}}$ in powers of $d\phi_{xy} \, d\bar{\phi}_{xy}$; likewise $e^{\sum B \, d\phi \, d\bar{\phi}}$ in terms of $d\phi_x \, d\bar{\phi}_x$. Argue that the terms in this expansion are naturally labelled by pairs (F, R) (Forest, roots).