EXERCISES for PIMS Course on Stochastic Poulation Systems

PART I - BGW Branching Processes

1. Let $\{Z_n\}$ be a BGW process with PGF $f(z)$ and $Z_0 = 1$. Assume that $m \leq 1$. Find the PGF of

$$Z_0 + Z_1 + \dots,$$

2. Consider the supercritical BGW process with offspring distribution with finite second moment. Prove directly (not using the MCT) that

$$\frac{Z_n}{m^n}$$

is a Cauchy sequence in L^2.

3. Consider the supercritical BGW process with offspring distribution with offspring distribution $\{p_k\}$ and finite second moment. Is there an invariance principle, that is, for fixed $m > 1$ and offspring variance σ^2 is the distribution of W independent $\{p_k\}$? - Justify your answer.

4. Let Z_n be a BGW process with $p_1 < 1$. Prove that

$$P(\lim_{n\to\infty} Z_n \in \{0, \infty\}) = 1.$$

5. Let $m = 1, \sigma^2 < \infty$. Show that

$$\lim_{n\to\infty} n^2[f_{n+1}(s) - f_n(s)] = \frac{2}{\sigma^2}, \quad s < 1$$

PART II Critical Branching Processes

1. Show that the birth and death process can be realized on a probability space as follows $(\Omega, \mathcal{F}, P)$ on which independent Poisson random measures N_1, N_2 on $\mathbb{R}_+^2$ are defined. Then the birth and death process is defined via a stochastic differential equation driven by the Poisson noises, namely,

$$X_t = x_0 + \int_0^t \int_0^{bX(s_-)} N_1(du, ds) - \int_0^t \int_0^{dX(s-)} N_2(du, ds). \tag{1}$$

This equation has a pathwise unique càdlàg solution which is a continuous time Markov chain with the required transition rates.

2. Generalize the linear birth and death process as follows. When a particle dies it produces k offspring with probability p_k, $k = 0, 1, 2, \dots$. Obtain a partial differential for the Laplace transform $L(t, \theta)$.

3. Let $X(t)$ be a critical Feller CSB process starting with $X(0) = 1$. Find the Laplace transform for the $Y(t)$ where $Y(t)$ is distributed as the size-biased law. Find the expected value at time t. Does $\frac{Y(t)}{E(Y(t)}$ converge in distribution?

4. Let $X_1(t)$ and $X_2(t)$ be Feller CSB with immigration satisfying

$$dX_i(t) = cdt + bX_i(t) + \sqrt{\gamma_i X_i(t)} dW_i(t)$$

where W_1, W_2 are independent Brownian motions. Derive the SDE's for the pair

$$(Y(t), Z(t))$$

where $Y(t) = X_1(t) + X_2(t),\ \ Z(t) = \frac{X_1(t)}{Y(t)}$ up to the time of extinction of $Y(t)$.

5. Prove that

$$d_{GH}(\mathcal{T}_g, \mathcal{T}_{g'}) \leq 2 \parallel g - g' \parallel . \quad (2)$$

PART III - Wright-Fisher and Infinitely many alleles processes

1. Consider the Kingman coalescent process, that is, the pure death process with death rate

$$d_k = k(k-1).$$

Prove that there exists an entrance law from infinity ("coming down from infinity") , that is, an integer-valued process with these death rates, $\{K(t) : t \geq 0\}$ such that

$$\lim_{t \downarrow 0} K(t) = \infty.$$

2. Construct the Kingman coalescent as a compact random metric space.

3. Let X be the stationary infinitely many alleles model with uniform mutation law. Extend its time parameter set to $(-\infty, \infty)$.

(a) Show that $\{X(t) : -\infty < t < \infty\}$ and $\{X(-t) : -\infty < t < \infty\}$ induce the same distribution on $C_{P([0,1])}(-\infty, \infty)$.

(b) Using (a), show that the probability that the most frequent allele at time 0, say, is oldest equals the probability that the most frequent allele at time 0 will survive the longest.

4. Show that the distribution of sizes of the age ordered alleles in the infinitely many alleles model is given by the GEM distribution. Use this to find the mean frequency of the jth oldest allele in the population.

PART IV - MISCELLANEOUS

1. Consider a two level branching process. The level two objects is a binary BGW process and each level two object consists of a number of level one objects. Within each level two object the level one objects form a binary BGW process. When a level two object produces offspring they begin with the number of particles in the parent object. Model this as a branching Markov chain. Obtain an equation for the log-Laplace equation.

2. Prepare an exposition on the Gromov-Hausdorff metric and metric measure spaces.