9.2 Branching random walk and branching Brownian motions

Branching random walks and branching diffusions have a long history. A general theory of branching Markov processes was developed in a series of three papers by Ikeda, Nagasawa and Watanabe in 1968, 1969 [327]. The application of branching random fields to genetics was introduced by Sawyer (1975) [534].

We consider a branching random walk (BRW). The dynamics are given by:

- Birth and death at rate γ:

$$\delta_x \to (k \text{ particles})\ \delta_x + \cdots + \delta_x \text{ w.p. } p_k, \quad \delta_x \to \emptyset \quad \text{w.p. } p_0,$$
$$\mathcal{G}(z) = \sum_{k=0}^{\infty} z^k p_k \quad \text{offspring distribution generating function.}$$

- Spatial random walk in S_1 with kernel $p(\cdot)$

$$\delta_x \to \quad \delta_y \text{ with rate } p(y-x)$$

The BRW is critical, subcritical, supercritical depending on $m = \sum_k kp_k = 1, < 1, > 1$, respectively.

We can write the generator of the branching rate walk as follows: $\mathcal{D} = \{F : F(\mu) = f(\mu(\phi)) = f(\langle \phi, \mu \rangle),\ \phi \in \mathcal{B}_b(S_1),\ f \in C(\mathbb{R})$ and for $F \in \mathcal{D}$,

$$\begin{aligned} GF(\mu) = \quad & \sum_x \mu(x) \sum_y p(y)[F(\mu + \delta_{x+y} - \delta_x) - F(\mu)] \\ & + \kappa \sum_x \mu(x) \sum_{k=0}^{\infty} p_k[F(\mu + (k-1)\delta_x) - F(\mu)] \\ & = \sum_x \mu(x) \sum_y p(y)[f(\mu(\phi) + \phi(x+y) - \phi(x)) - f(\mu(\phi))] \\ & \kappa \sum_x \mu(x) \sum_{k=0}^{\infty} p_k[f(\mu(\phi) + (k-1)\phi(x)) - f(\mu(\phi))] \end{aligned}$$

Let $\{S_t : t \geq 0\}$ denote the semigroup acting on $\mathcal{B}_b(S_1)$ associated to the random walk. Now define the Laplace functional

(9.6) $u(t,x) = P_{\delta_x}(e^{-X_t(\phi)}).$

Then conditioning at the first birth-death event we obtain

(9.7) $u(t,x) = (S_t e^{-\phi})(x) e^{-\kappa t} + \kappa \int_0^t e^{-\kappa s} (S_s \mathcal{G}(u(s,\cdot)))(x) ds.$

Note that this is also valid if we replace the random walk by a Lévy process on a locally compact abelian group.

Proposition 9.7 *The martingale problem for G is well posed and the Laplace functional of the solution is the unique solution of equation (9.7).*

The system of branching Brownian motions (BBM) is defined in the same way with $S = \mathbb{R}^d$ with offspring produced at the location of the parent and between branching the particles perform independent Brownian motions. (For non-local branching see Z. Li [429].)

Remark 9.8 *We sometimes combine the reproduction and spatial jump by replacing the reproduction and migration by a single mechanism in which an offspring produced by a birth immediately moves to a new location obtained by taking a jump with kernel p_ε, that is, $\delta_x \to \delta_x + \delta_y$.*

We also consider the $\mathcal{N}(S)$-measure-valued process $\{X_t\}$ in which each particle has mass η, that is,

$$X_t(A) = \eta \sum_{i=1}^{N(t)} \delta_{x_i(t)}, \quad A \subset S$$

where $x_i(t)$ denotes the location of the ith particles at time t.

Supercritical BRW and BBM

There is an important relation between supercritical branching Brownian motions and the Fisher-KPP equation. This relation was developed by McKean [473] and Bramson [49].

A basic question concerns the geometrical properties of the supercritical branching random walk. Biggins [38] has proved that the set $\mathcal{I}^{(n)}$ of positions occupied by nth generation individuals rescaled by a factor $\frac{1}{n}$ has asymptotic shape $\mathcal{I}$ where $\mathcal{I}$ is a convex set.

9.5 Measure-valued branching processes

9.5.1 Super-Brownian motion

Introduction

Super-Brownian motion (SBM) is a measure-valued branching process which generalizes the Jirina process. It was constructed by S. Watanabe (1968) [595] as a continuous state branching process and Dawson (1975) [113] in the context of SPDE. The lecture notes by Dawson (1993) [139] and Etheridge (2000) [217] provide introductions to measure-valued processes. The books of Dynkin [204], [205], Le Gall [424], Perkins [514] and Li [434] provide comprehensive developments of various aspects of measure-valued branching processes. In this section we begin with a brief introduction and then survey some aspects of superprocesses which are important for the study of stochastic population models. Section 9.5 gives a brief survey of the small scale properties of SBM and Chapter 10 deals with the large space-time scale properties.

Of special note is the discovery in recent years that super-Brownian motion arises as the scaling limit of a number of models from particle systems and statistical physics. An introduction to this class of *SBM invariance principles* is presented in Section 9.6 with emphasis on their application to the voter model and interacting Wright-Fisher diffusions. A discussion of the invariance properties of Feller CSB in the context of a renormalization group analysis is given in Chapter 11.

The SBM Martingale Problem

Let $(D(A), A)$ be the generator of a Feller process on a locally compact metric space (E, d) and $\gamma \geq 0$. The probability laws $\{P_\mu : \mu \in M_f(E)\}$ on $C([0,\infty), M_f(E))$ of the superprocess associated to (A, a, γ) can be characterized as the unique solution of the following martingale problem:

$$M_t(\varphi) := \langle \varphi, X_t \rangle - \int_0^t \langle A\varphi, X_s \rangle \, ds$$

is a P_μ-martingale with increasing process

$$\langle M(\varphi) \rangle_t = \int_0^t \gamma \, \langle \varphi, X_s \rangle \, ds$$

for each $\varphi \in D(A)$.

Equivalently, it solves the martingale problem

$$\begin{aligned} GF &= \int A\frac{\delta F}{\delta\mu(x)}\mu(dx) \\ &\quad +\frac{\gamma}{2}\iint \frac{\delta^2 F}{\delta\mu(x)\delta\mu(y)}\delta_x(dy)\mu(dx) \end{aligned}$$

$$D(G) := \{F(\mu) = e^{-\mu(\varphi)},\ \varphi \in \mathcal{B}_+(\mathbb{R}^d)\}$$

The special case $E = [0,1]$, $Af(x) = [\int f(y)\nu_0(dy) - f(x)]dy$ is the Jirina process. The special case $E = \mathbb{R}^d$ $A = \frac{1}{2}\Delta$ on $D(A) = C_b^2(\mathbb{R}^d)$ is called super-Brownian motion.

9.5.2 Super-Brownian Motion as the Limit of Branching Brownian Motion

Given a system of branching Brownian motions on $S = \mathbb{R}^d$ and $\varepsilon > 0$ we consider the measure-valued process, X^ε, with particle mass $m_\varepsilon = \varepsilon$ and branching rate $\gamma_\varepsilon = \frac{\gamma}{\varepsilon}$, that is,

$$X^\varepsilon(t) = m_\varepsilon \sum_{j=1}^{N(t)} \delta_{x_j(t)} \tag{9.32}$$

where $N(t)$ denotes the number of particles alive at time t and $x_1(t), \ldots, x_{N(t)}$ denote the locations of the particles at time t. Given an initial set of particles, let $\mu_\varepsilon = m_\varepsilon \sum_{j=1}^{N(0)} \delta_{x_j(0)}$, let $P^\varepsilon_{\mu_\varepsilon}$ denote the probability law of X^ε on $D_{M_F(\mathbb{R}^d)}([0,\infty))$. Let $\{\mathcal{F}_t\}_{t\ge 0}$ be the canonical filtration on $D([0,\infty), M_F(\mathbb{R}^d))$.

Notation 9.13 $\mu(\phi) = \langle\phi,\mu\rangle = \int \phi d\mu$.

Let $C(M_F(\mathbb{R}^d)) \supset D(G_\varepsilon) := \{F(\mu) = f(\langle\phi,\mu\rangle)) : f \in C_b^2(\mathbb{R}),\ \phi \in C_b^2(\mathbb{R}^d)\}$. Then $D(G_\varepsilon)$ is measure-determining on $M_F(\mathbb{R}^d)$ ([139], Lemma 3.2.5.).

Then using Itô's Lemma, it follows that $P^\varepsilon_{\mu_\varepsilon} \in \mathcal{P}(D([0,\infty), M_F(\mathbb{R}^d)))$ satisfies the G^ε-martingale problem where for $F \in D(G^\varepsilon)$,

$$\begin{aligned} G^\varepsilon F(\mu) &= f'(\mu(\phi))\mu(\frac{1}{2}\Delta\phi) + \frac{\varepsilon}{2}f''(\mu(\phi))\mu(\nabla\phi\cdot\nabla\phi) \\ &\quad +\frac{\gamma}{2\varepsilon^2}\int [f(\mu(\phi)+\varepsilon\phi(x)) + f(\mu(\phi)-\varepsilon\phi(x)) - 2f(\mu(\phi))]\mu(dx). \end{aligned}$$

Theorem 9.14 *Assume that $X^\varepsilon(0) = \mu_\varepsilon \Rightarrow \mu$ as $\varepsilon \to 0$.*

Then

(a) $P^\varepsilon_{\mu_\varepsilon} \overset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}_\mu \in \mathcal{P}(C_{M_F(\mathbb{R}^d)}([0,\infty))$ and $\mathbb{P}_\mu$ is the unique solution to the martingale problem: for all $\phi \in C_b^2(\mathbb{R}^d)$,

$$M_t(\phi) := X_t(\phi) - \mu(\phi) - \int_0^t X_s(\frac{1}{2}\Delta\phi)ds \tag{9.33}$$

is an $(\mathcal{F}_t^X)-$*martingale starting at zero with increasing process*

$$\langle M(\phi)\rangle_t = \gamma \int_0^t X_s(\phi^2)ds.$$

(b) The Laplace functional of X_t *is given by*

$$\mathbb{P}_\mu \left(e^{\left(-\int \phi(x)\mathbb{X}_t(dx)\right)}\right) = e^{\left(-\int v_t(x)\mu(dx)\right)}. \tag{9.34}$$

where $v(t,x)$ *is the unique solution of*

$$\frac{\partial v(t,x)}{\partial t} = \frac{1}{2}\Delta v(t,x) - \frac{\gamma}{2}v^2(t,x), \quad v_0 = \phi \in C^2_{+,b}(\mathbb{R}^d). \tag{9.35}$$

(c) The total mass process $\{X_t(\mathbb{R}^d\}_{t\geq 0}$ *is a Feller CSBP.*

Proof.

Step 1. Tightness of probability laws of X^ε on $D_{M_F(\mathbb{R}^d)}([0,\infty$ and a.s. continuity of limit points. In order to prove tightness it suffices to prove that for $\delta > 0$ there exists a compact subset $K \subset \mathbb{R}^d$ and $0 < L < \infty$ such that

(9.36)
$$P^\varepsilon_{\mu_\varepsilon}(\sup_{0\leq t\leq T} X_t(K^c) > \delta) < \delta, \;\; P^\varepsilon_{\mu_\varepsilon}(\sup_{0\leq t\leq T} X_t(1) > L) < \delta$$

and

(9.37) $P^\varepsilon_{\mu_\varepsilon} \circ (X_t(\phi))^{-1}$ is tight in $D_{\mathbb{R}}([0,\infty))$ for $\phi \in C^2_c(\mathbb{R}^d)$.

This can be checked by standard moment and martingale inequality arguments. For example for (9.36) it suffices to show that

$$\sup_{0<\varepsilon\leq 1}\sup_{\delta>0} E(\sup_{0\leq t\leq T}\langle e^{-\delta\|x\|}(1+\|x\|^2), \mathbf{X}^\varepsilon(t)\rangle) < \infty, \tag{9.38}$$

and (9.37) can be verified using the Joffe-Métivier criterion (see Appendix, (17.4.2)). The a.s. continuity of any limit point then follows from Theorem 17.14 since the maximum jump size in X^ε is ε.

Moreover, if $\mathbb{P}_\mu$ is a limit point, it is also easy to check (cf. Lemma 16.2) that for ϕ in $C^2_b(\mathbb{R}^d)$, $M_t(\phi)$ is a $\mathbb{P}_\mu$-martingale and $(F_1(\mu) = \mu(\phi),\; F_2(\mu) = \mu(\phi)^2)$

$$\begin{aligned}\langle M(\phi)\rangle_t &= \lim_{\varepsilon\to 0}\int_0^t (G_\varepsilon F_2(X_s) - 2F_1(X_s)G_\varepsilon F_1(X_s))ds \\ &= \gamma\int_0^t X_s(\phi^2)ds.\end{aligned}$$

As pointed out above, (9.33) and Ito's formula yields an equivalent formulation