Chapter 7

Martingale Problems and Dual Representations

7.1 Introduction

We have introduced above the basic mechanisms of branching, resampling and mutation mainly using generating functions and semigroup methods. However these methods have limitations and in order to work with a wider class of mechanisms we will introduce some this additional tools of stochastic analysis in this chapter. The martingale method which we use has proved to be a natural framework for studying a wide range of problems including those of population systems. The general framework is as follows:

- the object is to specify a Markov process on a Polish space E in terms of its probability laws $\{P_x\}_{x\in E}$ where P_x is a probability measure on $C_E([0,\infty))$ or $D_E([0,\infty))$ satisfying $P_x(X(0)=x)=1$.
- the probabilities $\{P_x\} \in \mathcal{P}(C_E([0,\infty)))$ satisfy a *martingale problem* (MP). One class of martingale problems is defined by the set of conditions of the form

  (7.1) $$F(X(t)) - \int_0^t GF(X(s))ds, \quad F \in \mathcal{D} \qquad (\mathcal{D}, G) - \text{martingale problem}$$

  is a P_x martingale where G is a linear map from $\mathcal{D}$ to $C(E)$, and $\mathcal{D} \subset C(E)$ is measure-determining.
- the martingale problem MP has one and only one solution.

Two martingale problems MP_1, MP_2 are said to be *equivalent* if a solution to MP_1 problem is a solution to MP_2 and vice versa.

In our setting the existence of a solution is often obtained as the limit of a sequence of probability laws of approximating processes. The question of uniqueness is often the more challenging part. We introduce the method of dual representation which can be used to establish uniqueness for a number of basic population processes. However the method of duality is applicable only for special classes of models. We introduce a second method, the Cameron-Martin-Girsanov type change of measure which is applicable to some basic problems of stochastic population systems. Beyond the domain of applicability of these methods, things are much more challenging. Some recent progress has been made in a series of papers of Athreya, Barlow, Bass, Perkins [11], Bass-Perkins [28], [29] but open problems remain.

We begin by reformulating the Jirina and neutral IMA Fleming-Viot in the martingale problem setting. We then develop the Girsanov and duality methods in the framework of measure-valued processes and apply them to the Fleming-Viot process with selection.

7.2 The Jirina martingale problem

By our projective limit construction of the Jirina process (with ν_0 = Lebesgue), we have a probability space $(\Omega, \mathcal{F}, \{X^\infty : [0,\infty) \times \mathcal{B}([0,1]) \to [0,\infty)\}, P)$ such that a.s. $t \to X^\infty_t(A)$ is continuous and $A \to X^\infty_\cdot(A)$ is finitely additive. We can take a modification, X, of X^∞ such that a.s. $X : [0,\infty) \to M_F([0,1])$ is continuous where $M_F([0,1])$ is the space of (countably additive) finite measures on $[0,1]$ with the weak topology. We then define the filtration

$$\mathcal{F}_t : \sigma\{X_s(A) : 0 \le s \le t,\ A \in \mathcal{B}([0,1])\}$$

and $\mathcal{P}$, the σ-field of predictable sets in $\mathbb{R}_+ \times \Omega$ (ie the σ-algebra generated by the class of $\mathcal{F}_t$-adapted, left continuous processes).

Recall that for a fixed set A the Feller CSB with immigration satisfies

$$(7.2)\quad X_t(A) - X_0(A) - \int_0^t c(\nu_0(A) - X_s(A))ds = \int_0^t \sqrt{2\gamma X_t(A)}dw^A_t$$

which is an L^2-martingale.

Moreover, by polarization

$$(7.3)\quad \langle M(A_1), M(A_2)\rangle_t = \gamma \int_0^t X_s(A_1 \cap A_2)ds$$

and if $A_1 \cap A_2 = \emptyset$, then the martingales $M(A_1)_t$ and $M(A_2)_t$ are orthogonal. This is an example of an *orthogonal martingale measure.*

Therefore for any Borel set A

$$(7.4)\quad M_t(A) := X_t(A) - X_0(A) - \int_0^t c(\nu_0(A) - X_s(A))ds$$

is a martingale with increasing process

$$(7.5)\quad \langle M(A)\rangle_t = \gamma\int_0^t X_s(A)ds.$$

We note that we can define integrals with respect to an orthogonal martingale measure (see next subsection) and show that (letting $X_t(f) = \int f(x)X_t(dx)$ for $f \in \mathcal{B}([0,1])$)

$$(7.6)\quad M_t(f) := X_t(f) - X_0(f) - \int_0^t c(\nu_0(f) - X_s(f))ds = \int f(x)M_t(dx)$$

which is a martingale with increasing process

$$(7.7)\quad \langle M(f)\rangle_t = \gamma\int_0^t f^2(x)X_s(dx)ds.$$

This suggests the martingale problem for the Jirina process which we state in subsection 7.2.2.

7.2.1 Stochastic Integrals wrt Martingale Measures

A general approach to martingale measures and stochastic integrals with respect to martingale measures was developed by Walsh [597]. We briefly review some basic results.

Let

$$\mathcal{L}^2_{loc} = \left\{\psi : \mathbb{R}_+ \times \Omega \times E \to \mathbb{R} : \psi \text{ is } \mathcal{P}\times\mathcal{E}\text{-measurable}, \int_0^t X_s(\psi_s^2)ds < \infty,\ \forall t > 0\right\}$$

A $\mathcal{P}\times\mathcal{E}$-measurable function ψ is *simple* ($\psi \in \mathcal{S}$) iff

$$\psi(t,\omega,x) = \sum_{i=1}^K \psi_{i-1}(\omega)\phi_i(x)1_{(t_{i-1},t_i]}(t)$$

for some $\phi_i \in b\mathcal{B}([0,1])$, $\psi \in b\mathcal{F}_{t_{i-1}}$, $0 = t_0 < t_1 \cdots < t_K \leq \infty$. For such a ψ, define

$$M_t(\psi) := \int_0^t\int \psi(s,x)dM(s,x) = \sum_{i=1}^K \psi_{i-1}(M_{t\wedge t_i}(\phi_i) - M_{t\wedge t_{i-1}}(\phi_i))$$

Then $M_t(\psi) \in \mathcal{M}_{\text{loc}}$ (the space of $\mathcal{F}_t$ local martingales) and

$$\langle M(\psi)_t\rangle = \int_0^t X_s(\gamma\psi_s^2)ds.$$

Lemma 7.1 *For any $\psi \in \mathcal{L}^2_{loc}$ there is a sequence $\{\psi_n\}$ in $\mathcal{S}$ such that*

$$P\left(\int_0^n \int (\psi_n - \psi)^2(s,\omega,x)\gamma(x)X_s(dx)ds > 2^{-n}\right) < 2^{-n}.$$

Proof. Let $\bar{\mathcal{S}}$ denote the set of bounded $\mathcal{P}\times\mathcal{E}$-measurable functions which can be approximated as above. $\bar{\mathcal{S}}$ is closed under $\to^{bp}$. Using $\mathcal{H}_0 = \{f_{i-1}(\omega)\phi_i(x), \phi \in b\mathcal{E},\ f_{i-1} \in b\mathcal{F}_{t_{i-1}}, \phi_i \in b\mathcal{E}\}$, we see that $\psi(t,\omega,x) = \sum_{i=1}^K \psi_{i-1}(\omega,x)1_{(t_{i-1},t_i]}(t)$ is in $\bar{\mathcal{S}}$ for any $\psi_{i-1} \in b(\mathcal{F}_{t_{i-1}}\times\mathcal{E})$. If $\psi \in b(\mathcal{P}\times\mathcal{E})$, then

$$\psi_n(s,\omega,x) = 2^n \int_{(i-1)2^{-n}}^{i2^{-n}} \psi(r,\omega,x)dr \text{ is } s \in (i2^{-n},(i+1)2^{-n}],\ i=1,2,\dots$$

satisfies $\psi_n \in \bar{\mathcal{S}}$ by the above. For each $(\omega,x), \psi_n(s,\omega,x) \to \psi(s,\omega,x)$ for Lebesgue a.a. s by Lebesgue's differentiation theorem and it follows easily that $\psi \in \bar{\mathcal{S}}$. Finally if $\psi \in \mathcal{L}^2_{\text{loc}}$, the obvious truncation argument and dominated convergence (set $\psi_n = (\psi \wedge n) \vee (-n)$ completes the proof. ∎

Proposition 7.2 *There is a unique linear extension of $M : \mathcal{S} \to \mathcal{M}_{loc}$ (the space of local martingales) to a map $M : \mathcal{L}^2_{loc} \to \mathcal{M}_{loc}$ such that $M_t(\psi)$ is a local martingale with increasing process $\langle M(\psi)\rangle_t$ given by*

$$\langle M(\psi)\rangle_t := \int_0^t \gamma X_s(\psi_s^2)ds\ \forall\, t \geq 0\ \ a.s. \forall\, \psi \in \mathcal{L}^2_{loc}.$$

Proof. We can choose $\psi_n \in \mathcal{S}$ as in the Lemma. Then

$$\langle M(\psi) - M(\psi_n)\rangle_n = \langle M(\psi - \psi_n)\rangle_n = \gamma\int_0^n X_s(\gamma(\psi(s) - \psi_n(s))^2 ds$$
$$P\left(\langle M(\psi) - M(\psi_n)\rangle_n > 2^{-n}\right) < 2^{-n}$$

The $\{M_t(\psi_n)\}_{t\geq 0}$ is Cauchy and using Doob's inequality and the Borel-Cantelli Lemma we can define $\{M_t(\psi)\}_{t\geq 0}$ such that

$$\sup_{t\leq n} |M_t(\psi) - M_t(\psi_n)| \to 0 \text{ a.s. as } n \to \dot{\infty}.$$

This yields the required extension and its uniqueness. ∎

Note that it immediately follows by polarization that if $\psi, \phi \in \mathcal{L}^2_{\text{loc}}$,

$$\langle M(\phi), M(\psi)\rangle_t = \gamma\int_0^t X_s(\phi_s\psi_s)ds$$

Moreover, in this case $M_t(\psi)$ is a L^2-martingale, that is,

$$E(\langle M(\psi)\rangle_t) = \gamma\int_0^t E(X_s(\psi_s^2))ds < \infty$$

provided that

$$\psi \in \mathcal{L}^2 = \{\psi \in \mathcal{L}^2_{\text{loc}} : E(\int_0^t X_s(\psi_s^2)ds < \infty,\ \forall t > 0\}.$$

Remark 7.3 *Walsh (1986) [597] defined a more general class of martingale measures on a measurable space $(E, \mathcal{E})$ for which the above construction of stochastic integrals can be extended. $\{M_t(A) : t \geq 0,\ A \in \mathcal{E}\}$ is an L^2-martingale measure wrt $\mathcal{F}_t$ iff*

(a) $M_0(A) = 0\ \ \forall A \in \mathcal{E}$,
(b) $\{M_t(A), t \geq 0\}$ is an $\mathcal{F}_t$-martingale for every $A \in \mathcal{E}$,
(c) for all $t > 0$, M_t is an L^2-valued σ-finite measure.

The martingale measure is worthy if there exists a σ-finite "dominating measure" $K(\cdot, \cdot, \cdot, \omega)$, on $\mathcal{E} \times \mathcal{E} \times \mathcal{B}(\mathbb{R}_+)$, $\omega \in \Omega$ such that

(a) K is symmetric and positive definite, i.e. for any $f \in b\mathcal{E} \times \mathcal{B}(\mathbb{R}_+)$,

$$\int\int\int f(x,s)f(y,s)K(dx,dy,ds) \geq 0$$

(b) for fixed A, B, $\{K(A \times B \times (0,t]), t \geq 0\}$ is $\mathcal{F}_t$-predictable
(c) $\exists\ E_n \uparrow E$ such that $E\{K(E_n \times E_n \times [0,T]\} < \infty\ \forall n$,
(d) $|\langle M(A), M(A)\rangle_t| \leq K(A \times A \times [0,t])$.

7.2.2 Uniqueness and stationary measures for the Jirina Martingale Problem

A probability law, $\mathbb{P}_\mu \in C_{M_F([0,1])}([0,\infty))$, is a solution of the *Jirina martingale problem*, if under P_μ, $X_0 = \mu$ and

$$(7.8)\quad \begin{aligned} &M_t(\phi) := X_t(\phi) - X_0(\phi) - \int_0^t c(\nu_0(\phi) - X_s(\phi))ds, \\ &\qquad \text{is a } L^2,\ \mathcal{F}_t\text{-martingale } \forall\ \phi \in b\mathcal{B}([0,1]) \text{ with increasing process} \\ &\langle M(\phi)\rangle_t = \gamma\int_0^t X_s(\phi^2)ds, \text{ that is,} \\ &\qquad M_t^2(\phi) - \langle M(\phi)\rangle_t \text{ is a martingale.} \end{aligned}$$

Remark 7.4 *This is* equivalent *to the martingale problem*

$$(7.9)\quad M_F(t) = F(X_t) - \int_0^t GF(X(s))ds \quad \textit{is a martingale}$$

for all $F \in \mathcal{D} \subset C(M_F([0,1])$ where

$$\mathcal{D} = \{F : F(\mu) = \prod_{i=1}^n \mu(f_i),\ \ f_i \in C([0,1]),\ i = 1, \ldots, , n,\ n \in \mathbb{N}\}$$

and

$$\begin{aligned} GF(\mu) = \quad & c\int\left[\int \frac{\partial F(\mu)}{\partial \mu(x)}\nu_0(dx) - \frac{\partial F(\mu)}{\partial \mu(x)}\right]\mu(dx) \\ & +\frac{\gamma}{2}\int\int \frac{\partial^2 F(\mu)}{\partial\mu(x)\partial\mu(y)}(\delta_x(dy)\mu(dx)-\mu(dx)\mu(dy)) \end{aligned}$$

Theorem 7.5 *There exists one and only one solution $P_\mu \in \mathcal{P}(C_{M_F([0,1])}([0,\infty))$ to the martingale problem (7.8). This defines a continuous $M_F(0,1])$ continuous strong Markov process.*
(b) (Ergodic Theorem) Given any initial condition, X_0, the law of X_t converges weakly to a limiting distribution as $t \to \infty$ with Laplace functional

$$(7.10)\quad E(e^{-\int_0^1 f(x)X_\infty(dx)}) = \exp\left(-\frac{2c}{\gamma}\int_0^1 \log(1+\frac{f(x)}{\theta})\nu_0(dx)\right).$$

This can be represented by

$$(7.11)\quad X_\infty(A) = \frac{1}{\theta}\int_0^1 f(x)G(\theta ds)$$

where G is the Gamma (Moran) subordinator (recall (6.17)).

Proof. Outline of method. As discussed above the projective limit construction produced a solution to this martingale problem.

A fundamental result of Stroock and Varadhan ([571] Theorem 6.2.3) is that in order to prove that the martingale problem has at most one solution it suffices to show that the one-dimensional marginal distributions $\mathcal{L}(X_t), t \geq 0$, are uniquely determined. Moreover in order to determine the law of a random measure, X, on $[0,1]$ it suffices to determine the Laplace functional.

The main step of the proof is to verify that if P_μ is a solution to the Jirina martingale problem, $t > 0$, and $f \in C_+([0,1])$, then

$$(7.12)\quad E_\mu(e^{-X_t(f)}) = e^{-\mu(\psi(t))-c\nu_0(\int_0^t \psi(t-s)ds)}$$

where

$$\begin{aligned} \frac{d\psi(s,x)}{ds} &= -c\psi(s,x) - \frac{\gamma}{2}\psi^2(s,x), \\ \psi(0,x) &= f(x). \end{aligned}$$

STEP 1: -discretization
We can choose a sequence of partitions $\{A_1^n,\dots,A_{K_n}^n\}$ and $\lambda_1^n,\dots,\lambda_{K_n}^n$ such that

$$(7.13)\quad \sum_{i=1}^{K_n}\lambda_i^n 1_{A_i^n} \uparrow f(\cdot).$$

We next show that for a partition $\{A_1,\dots,A_K\}$ of $[0,1]$ and $\lambda_i \geq 0,\ i=1,\dots,K$,

$$\exp(-\sum_{i=1}^{K}\lambda_i X_t(A_i)) = \exp\left(-\sum_{i=1}^{K}\psi_i(t)X_0(A_i) - \sum c\nu_0(A_i)\int_0^t \psi_i(t-s)ds\right)$$

$$\frac{d\psi_i}{ds} = -c\psi_i - \frac{\gamma}{2}\psi_i^2$$

$$\psi_i(0) = \lambda_i.$$

To verify this first note that by Itô's Lemma, for fixed t and $0 \leq s \leq t$,

$$\begin{aligned} d\psi_i(t-s)X_s(A_i) &= X_s(A_i)d\psi_i(t-s) + \psi_i(t-s)dX_s(A_i) \\ &= X_s(A_i)d\psi_i(t-s) + \psi_i(t-s)c\nu_0(A_i) \\ &\quad - c\psi_i(t-s)X_s(A_i) + \psi_i(t-s)dM_s(A_i) \end{aligned}$$

and

$$\begin{aligned} &X_t(A_i)\psi_i(0) - X_0(A_i)\psi_i(t) \\ &= -\int_0^t X_s(A_i)\dot{\psi}_i(t-s)ds + c\nu_0(A_i)\int_0^t \psi_i(t-s)ds \\ &\quad - c\int_0^t X_s(A_i)\psi_i(t-s)ds + N_t(A_i) \end{aligned}$$

where $\{N\}_{0\leq s\leq t}$ is an orthogonal martingale measure with

$$\begin{aligned} N_s(A_i) &= \int_0^s \psi_i(t-u)M(A_i,du) \\ \langle N(A_i)\rangle_s &= \frac{\gamma}{2}\int_0^s \psi_i^2(t-u)X_u(A_i)du \\ \langle N(A_i),N(A_j)\rangle_s &= 0 \ \text{ if } \ i\neq j. \end{aligned}$$

Again using Itô's lemma, for $0 \leq s \leq t$

$$\begin{aligned} de^{-X_s(A_i)\psi_i(t-s)} &= \dot{\psi}_i(t-s)e^{-X_s(A_i)\psi_i(t-s)}ds - \psi_i(t-s)e^{-X_s(A_i)\psi_i(t-s)}dX_s(A_i) \\ &\quad + \frac{\gamma}{2}e^{-X_s(A_i)\psi_i(t-s)}\psi_i^2(t-s)X_s(A_i)ds \\ &= \dot{\psi}_i(t-s)e^{-X_s(A_i)\psi_i(t-s)}ds + c\psi_i(t-s)e^{-X_s(A_i)\psi_i(t-s)}X_s ds \\ &\quad + c\nu_0(A)\psi_i(t-s)e^{-X_s(A_i)\psi_i(t-s)}ds \\ &\quad + \frac{\gamma}{2}e^{-X_s(A_i)\psi_i(t-s)}\psi_i^2(t-s)X_s ds + dN_s(A_i) \\ &= c\nu_0(A_i)\psi_i(t-s)e^{-X_s(A_i)\psi_i(t-s)}ds + dN_s(A_i) \end{aligned}$$

Then by the method of integrating factors we can get

$$\widetilde{N}_s(A_i) = e^{\left(-X_s(A_i)\psi_i(t-s)+c\nu_0(A_i)\int_s^t \psi_i(t-u)du\right)}, \quad 0 \le s \le t,$$

is a bounded non-negative martingale that can be represented as

$$(7.14)\ \ \widetilde{N}_t(A_i) - \widetilde{N}_0(A_i) = \int_0^t e^{-\zeta_i(s)} dN_s(A_i).$$

where

$$(7.15)\ \ \zeta_i(s) = \left(c\nu_0(A_i)\int_s^t \psi_i(t-u)du\right).$$

Noting that the martingales $\widetilde{N}_t(A_i), \widetilde{N}_t(A_j)$ are orthogonal if $i \ne j$ we can conclude that

$$(7.16)\ \ e^{-\sum_i\left(X_s(A_i)\psi_i(t-s)-c\nu_0(A_i)\int_s^t \psi_i(t-u)du\right)}, \quad 0 \le s \le t,$$

is a bounded martingale. Therefore for each n

$$(7.17)\ \ E\left[e^{-\sum_{i=1}^{K_n}\left(X_t(A_i^n)\psi_i^n(0)\right)}\right] = e^{-\sum_{i=1}^{K_n}\left(X_0(A_i^n)\psi_i^n(t)-c\nu_0(A_i^n)\int_s^t \psi_i^n(t-u)du\right)}$$

STEP 2: Completion of the proof
Taking limits as $n \to \infty$ and dominated convergence we can then show that the Laplace functional

$$E(e^{-X_t(f)}) = e^{-X_0(\psi(t))-c\nu_0(\int_0^t \psi(t-s)ds)}$$

where

$$\frac{d\psi(s,x)}{ds} = -c\psi(s,x) - \frac{\gamma}{2}\psi^2(s,x),$$
$$\psi(0,x) = f(x).$$

Therefore the distribution of $X_t(f)$ is determined for any non-negative continuous function, f, on $[0,1]$. Since the Laplace functional characterizes the law of a random measure, this proves that the distribution at time t is uniquely determined by the martingale problem. This completes the proof of uniqueness.

(b) Recall that $\psi(\cdot,\cdot)$ satisfies

$$\frac{d\psi(x,s)}{ds} = -c\psi(x,s) - \frac{\gamma}{2}\psi^2(x,s),$$
$$\psi(x,0) = f(x)$$

Solving, we get

$$\psi(x,t) = \frac{f(x)e^{-ct}}{1 + \frac{f(x)}{\theta} - \frac{f(x)}{\theta}e^{-ct}}, \ \theta = \frac{2c}{\gamma}$$

Next, note that $\psi(x,t) \to 0$ as $t \to \infty$ and

$$\begin{aligned}
\int_0^\infty \psi(x,s)ds &= \int_0^\infty \frac{f(x)e^{-ct}}{1 + \frac{f(x)}{\theta} - \frac{f(x)}{\theta}e^{-ct}} dt = \int \frac{(-\frac{f(x)}{c})d(e^{-ct})}{1 + \frac{f(x)}{\theta} - \frac{f(x)}{\theta}(e^{-ct})} \\
&= \frac{2}{\gamma}\int_0^1 \frac{\frac{f(x)}{\theta}du}{1 + \frac{f(x)}{\theta} - \frac{f(x)}{\theta}u} = \frac{2}{\gamma}\int_0^{\frac{f(x)}{\theta}} \frac{\frac{f(x)}{\theta}du}{1 + \frac{f(x)}{\theta} - \frac{f(x)}{\theta}u} \\
&= \frac{2}{\gamma}\log(1 + \frac{f(x)}{\theta})
\end{aligned}$$

Therefore

$$E(e^{-X_t(f)}) \to e^{-\frac{2c}{\gamma}\int \log(1+\frac{f(x)}{\theta})\nu_0(dx)}$$

This coincides with the Laplace functional of

$$\frac{1}{\theta}\int f(s)G(\theta ds)$$

where $G(\cdot)$ is the Moran subordinator. Therefore $X_\infty(f)$ can be represented as

$$X_\infty(f) = \frac{1}{\theta}\int f(s)G(\theta ds).$$

■

Remark 7.6 *A more general class of measure-valued branching processes, known as* superprocesses *or* Dawson-Watanabe processes *will be discussed in Section 9.4.*

9.5 Measure-valued branching processes

9.5.1 Super-Brownian motion

Introduction

Super-Brownian motion (SBM) is a measure-valued branching process which generalizes the Jirina process. It was constructed by S. Watanabe (1968) [596] as a continuous state branching process and Dawson (1975) [114] in the context of SPDE. The lecture notes by Dawson (1993) [140] and Etheridge (2000) [218] provide introductions to measure-valued processes. The books of Dynkin [205], [206], Le Gall [425], Perkins [515] and Li [435] provide comprehensive developments of various aspects of measure-valued branching processes. In this section we begin with a brief introduction and then survey some aspects of superprocesses which are important for the study of stochastic population models. Section 9.5 gives a brief survey of the small scale properties of SBM and Chapter 10 deals with the large space-time scale properties.

Of special note is the discovery in recent years that super-Brownian motion arises as the scaling limit of a number of models from particle systems and statistical physics. An introduction to this class of *SBM invariance principles* is presented in Section 9.6 with emphasis on their application to the voter model and interacting Wright-Fisher diffusions. A discussion of the invariance properties of Feller CSB in the context of a renormalization group analysis is given in Chapter 11.

The SBM Martingale Problem

Let $(D(A), A)$ be the generator of a Feller process on a locally compact metric space (E, d) and $\gamma \geq 0$. The probability laws $\{P_\mu : \mu \in M_f(E)\}$ on $C([0,\infty), M_f(E))$ of the superprocess associated to (A, a, γ) can be characterized as the unique solution of the following martingale problem:

$$M_t(\varphi) := \langle \varphi, X_t \rangle - \int_0^t \langle A\varphi, X_s \rangle \, ds$$

is a P_μ-martingale with increasing process

$$\langle M(\varphi) \rangle_t = \int_0^t \gamma \, \langle \varphi, X_s \rangle \, ds$$

for each $\varphi \in D(A)$.

Equivalently, it solves the martingale problem

$$\begin{aligned} GF &= \int A\frac{\delta F}{\delta\mu(x)}\mu(dx) \\ &\quad +\frac{\gamma}{2}\iint \frac{\delta^2 F}{\delta\mu(x)\delta\mu(y)}\delta_x(dy)\mu(dx) \end{aligned}$$

$$D(G) := \{F(\mu) = e^{-\mu(\varphi)},\ \varphi\in\mathcal{B}_+(\mathbb{R}^d)\}$$

The special case $E = [0,1]$, $Af(x) = [\int f(y)\nu_0(dy) - f(x)]dy$ is the Jirina process. The special case $E = \mathbb{R}^d$ $A = \frac{1}{2}\Delta$ on $D(A) = C_b^2(\mathbb{R}^d)$ is called super-Brownian motion.

9.5.2 Super-Brownian Motion as the Limit of Branching Brownian Motion

Given a system of branching Brownian motions on $S = \mathbb{R}^d$ and $\varepsilon > 0$ we consider the measure-valued process, X^ε, with particle mass $m_\varepsilon = \varepsilon$ and branching rate $\gamma_\varepsilon = \frac{\gamma}{\varepsilon}$, that is,

$$(9.32)\quad X^\varepsilon(t) = m_\varepsilon\sum_{j=1}^{N(t)}\delta_{x_j(t)}$$

where $N(t)$ denotes the number of particles alive at time t and $x_1(t),\dots,x_{N(t)}$ denote the locations of the particles at time t. Given an initial set of particles, let $\mu_\varepsilon = m_\varepsilon\sum_{j=1}^{N(0)}\delta_{x_j(0)}$, let $P^\varepsilon_{\mu_\varepsilon}$ denote the probability law of X^ε on $D_{M_F(\mathbb{R}^d)}([0,\infty))$. Let $\{\mathcal{F}_t\}_{t\geq 0}$ be the canonical filtration on $D([0,\infty), M_F(\mathbb{R}^d))$.

Notation 9.13 $\mu(\phi) = \langle\phi,\mu\rangle = \int\phi d\mu$.

Let $C(M_F(\mathbb{R}^d)) \supset D(G_\varepsilon) := \{F(\mu) = f(\langle\phi,\mu\rangle)) : f\in C_b^2(\mathbb{R}),\ \phi\in C_b^2(\mathbb{R}^d)\}$. Then $D(G_\varepsilon)$ is measure-determining on $M_F(\mathbb{R}^d)$ ([140], Lemma 3.2.5.).

Then using Itô's Lemma, it follows that $P^\varepsilon_{\mu_\varepsilon}\in\mathcal{P}(D([0,\infty), M_F(\mathbb{R}^d)))$ satisfies the G^ε-martingale problem where for $F\in D(G^\varepsilon)$,

$$\begin{aligned} G^\varepsilon F(\mu) &= f'(\mu(\phi))\mu(\frac{1}{2}\Delta\phi) + \frac{\varepsilon}{2}f''(\mu(\phi))\mu(\nabla\phi\cdot\nabla\phi) \\ &\quad +\frac{\gamma}{2\varepsilon^2}\int[f(\mu(\phi)+\varepsilon\phi(x)) + f(\mu(\phi)-\varepsilon\phi(x)) - 2f(\mu(\phi))]\mu(dx). \end{aligned}$$

We can also obtain the Laplace functional of $X^\varepsilon(t)$ using Proposition 9.7 with $\{S_t : t\geq 0\}$ the Brownian motion semigroup on $C(\mathbb{R}^d)$ and $\mathcal{G}(z) = \frac{1}{2}+\frac{1}{2}z^z$.

Theorem 9.14 *Assume that* $X^\varepsilon(0) = \mu_\varepsilon \Rightarrow \mu$ *as* $\varepsilon\to 0$.

Then
(a) $P^{\varepsilon}_{\mu_\varepsilon} \overset{\varepsilon\to 0}{\Longrightarrow} \mathbb{P}_\mu \in \mathcal{P}(C_{M_F(\mathbb{R}^d)}([0,\infty))$ *and* $\mathbb{P}_\mu$ *is the unique solution to the martingale problem: for all* $\phi \in C^2_b(\mathbb{R}^d)$,

$$(9.33)\quad M_t(\phi) := X_t(\phi) - \mu(\phi) - \int_0^t X_s(\frac{1}{2}\Delta\phi)ds$$

is an $(\mathcal{F}^X_t)-$*martingale starting at zero with increasing process*

$$\langle M(\phi)\rangle_t = \gamma\int_0^t X_s(\phi^2)ds.$$

(b) The Laplace functional of X_t *is given by*

$$(9.34)\quad \mathbb{P}_\mu\left(e^{\left(-\int\phi(x)\mathbb{X}_t(dx)\right)}\right) = e^{\left(-\int v_t(x)\mu(dx)\right)}.$$

where $v(t,x)$ *is the unique solution of*

$$(9.35)\quad \frac{\partial v(t,x)}{\partial t} = \frac{1}{2}\Delta v(t,x) - \frac{\gamma}{2}v^2(t,x), \quad v_0 = \phi \in C^2_{+,b}(\mathbb{R}^d).$$

(c) The total mass process $\{X_t(\mathbb{R}^d\}_{t\geq 0}$ *is a Feller CSBP.*

Proof.
Step 1. Tightness of probability laws of X^ε on $D_{M_F(\mathbb{R}^d)}([0,\infty$ and a.s. continuity of limit points. In order to prove tightness it suffices to prove that for $\delta > 0$ there exists a compact subset $K \subset \mathbb{R}^d$ and $0 < L < \infty$ such that

(9.36)
$$P^{\varepsilon}_{\mu_\varepsilon}(\sup_{0\leq t\leq T} X_t(K^c) > \delta) < \delta,\;\; P^{\varepsilon}_{\mu_\varepsilon}(\sup_{0\leq t\leq T} X_t(1) > L) < \delta$$

and

$$(9.37)\quad P^{\varepsilon}_{\mu_\varepsilon}\circ (X_t(\phi))^{-1} \text{ is tight in } D_{\mathbb{R}}([0,\infty)) \text{ for } \phi\in C^2_c(\mathbb{R}^d).$$

This can be checked by standard moment and martingale inequality arguments. For example for (9.36) it suffices to show that

$$(9.38)\quad \sup_{0<\varepsilon\leq 1}\sup_{\delta>0} E(\sup_{0\leq t\leq T}\langle e^{-\delta\|x\|}(1+\|x\|^2), \mathbf{X}^\varepsilon(t)\rangle) < \infty,$$

and (9.37) can be verified using the Joffe-Métivier criterion (see Appendix, (17.4.2)). The a.s. continuity of any limit point then follows from Theorem 17.14 since the maximum jump size in X^ε is ε.

Moreover, if $\mathbb{P}_\mu$ is a limit point, it is also easy to check (cf. Lemma 16.2) that for ϕ in $C^2_b(\mathbb{R}^d)$, $M_t(\phi)$ is a $\mathbb{P}_\mu$-martingale and $(F_1(\mu) = \mu(\phi),\; F_2(\mu) = \mu(\phi)^2)$

$$\begin{aligned}\langle M(\phi)\rangle_t &= \lim_{\varepsilon\to 0}\int_0^t (G_\varepsilon F_2(X_s) - 2F_1(X_s)G_\varepsilon F_1(X_s))ds\\ &= \gamma\int_0^t X_s(\phi^2)ds.\end{aligned}$$

As pointed out above, (9.33) and Ito's formula yields an equivalent formulation of the martingale problem, namely: for $f \in C_b^2(\mathbb{R})$, $\phi \in C_b^2(\mathbb{R}^d)$, and $F(\mu) = f(\mu(\phi))$,

(9.39) $$F(X_t) - \int_0^t GF(X_s)ds \text{ is a } P_\mu - martingale$$

where

$$GF(\mu) = f'(\mu(\phi))\mu(\frac{1}{2}\Delta\phi) + \frac{\gamma}{2} f''(\mu(\phi))\mu(\phi^2).$$

Step 2. (Uniqueness) In order to prove (b) we first verify that $\mathbb{P}_\mu$ also solves the following time dependent martingale problem. Let $\psi : [0,\infty) \times E \to [0,\infty)$ such that ψ, $\frac{\partial}{\partial s}\psi$ and $\Delta\psi$ are bounded and strongly continuous in $C_b(\mathbb{R}^d)$. Assume that

(9.40) $$\left\| \frac{\psi(s+h,\cdot) - \psi(s,\cdot)}{h} - \frac{\partial}{\partial s}\psi(s,\cdot) \right\|_\infty \to 0 \quad \text{as } h \to 0.$$

Then

(9.41)
$$\exp(-X_t(\psi_t)) + \int_0^t \exp(-X_s(\psi_s))X_s((A + \frac{\partial}{\partial s})\psi_s)ds - \frac{\gamma}{2}\int_0^t \exp(-X_s(\psi_s))X_s(\psi_s^2)ds$$

is a $\mathbb{P}_\mu$-martingale. Let $\mathbb{P}_\mu^{\mathcal{F}_t}$ denote the conditional expectation with respect to $\mathcal{F}_t$ under $\mathbb{P}_\mu$.

To prove (9.41) first note that applying (9.39) to $\exp(-\mu(\phi))$ with $\phi \in C_b^2(\mathbb{R}^d)$, we obtain

(9.42)
$$\mathcal{E}_t(\phi) = \exp(-X_t(\phi)) + \int_0^t \exp(-X_s(\phi))X_s(A\phi)ds - \frac{\gamma}{2}\int_0^t \exp(-X_s(\phi))X_s(\phi^2)ds$$

is a $\mathbb{P}_\mu$-martingale.

Next take

(9.43) $$u(s, X_t) = \exp(-X_t(\psi_s)), \quad v(s, X_t) = \exp(-(X_t(\psi_s))X_t(\frac{\partial}{\partial s}\psi_s), \text{ and}$$
$$w(s, X_t) = \exp(-X_t(\phi))(X_t(A\psi_s))$$

so that for $t_2 > t_1$

(9.44) $$u(t_2, X_{t_2}) - u(t_1, X_{t_2}) = -\int_{t_1}^{t_2} v(s, X_{t_2})ds.$$

Then using (9.42) we have

$$\mathbb{P}_\mu^{\mathcal{F}_{t_1}}[u(t_1,X_{t_2})-u(t_1,X_{t_1})] = -\mathbb{P}_\mu^{\mathcal{F}_{t_1}}[\int_{t_1}^{t_2} w(t_1,X_s)ds] + \frac{\gamma}{2}\mathbb{P}_\mu^{\mathcal{F}_{t_1}}[\int_{t_1}^{t_2} u(t_1,X_s)X_s(\psi_s^2)ds]. \tag{9.45}$$

Let Λ^n be a partition of $[t_1,t_2]$ with $\mathrm{mesh}(\Lambda^n)\to 0$ and

$$\begin{aligned}\psi^n(s,x) \;:&= \sum_{i=1}^n \psi(t_i^n,x)1_{[t_i^n,t_{i+1}^n)}(s)\\ X^n(s) \;:&= \sum_{i=1}^n X_{t_{i+1}^n}1_{[t_i^n,t_{i+1}^n)}(s)\end{aligned}$$

Let $u^n(t,X_t):=\exp(-X_t(\psi_t^n))$.

Then by (9.45)

$$\begin{aligned}\mathbb{P}_\mu^{\mathcal{F}_{t_1}}[u^n(t_2,X_{t_2})-u^n(t_1,X_{t_1})] &= -\mathbb{P}_m^{\mathcal{F}_{t_1}}[\int_{t_1}^{t_2}\exp(-X_s^n(\psi_s))X_s^n(\frac{\partial}{\partial s}\psi_s)ds]\\ &\quad -\mathbb{P}_\mu^{\mathcal{F}_{t_1}}[\int_{t_1}^{t_2}\exp(-X_s(\psi_s^n))X_s(A\psi_s^n)ds]\\ &\quad +\frac{\gamma}{2}\mathbb{P}_\mu^{\mathcal{F}_{t_1}}[\int_{t_1}^{t_2}\exp(-X_s(\psi_s^n))X_s((\psi_s^n)^2)ds].\end{aligned}$$

Standard arguments show that this converges to

$$\begin{aligned}\mathbb{P}_\mu^{\mathcal{F}_{t_1}}[u(t_2,X_{t_2})-u(t_1,X_{t_1})] &= -\mathbb{P}_m^{\mathcal{F}_{t_1}}[\int_{t_1}^{t_2}\exp(-X_s(\psi_s))X_s(\frac{\partial}{\partial s}\psi_s)ds]\\ &\quad -\mathbb{P}_\mu^{\mathcal{F}_{t_1}}[\int_{t_1}^{t_2}\exp(-X_s(\psi_s))X_s(A\psi_s)ds]\\ &\quad +\frac{\gamma}{2}\mathbb{P}_\mu^{\mathcal{F}_{t_1}}[\int_{t_1}^{t_2}\exp(-X_s(\psi_s))X_s((\psi_s)^2)ds]\end{aligned}$$

which completes the proof of (9.41).

Now let $v_t=V_t\phi$ be the unique solution (see [501]) of

$$\frac{\partial v_t}{\partial t} = Av_t - \frac{\gamma}{2}v_t^2,\quad v_0=\phi\in C^2_{+,b}(\mathbb{R}^d). \tag{9.46}$$

Then v_t satisfies (9.40). Applying (9.41) we deduce that $\{\exp^{(-X_s(v_{t-s}))}\}_{0\le s\le t}$ is a martingale. Equating mean values at $s=0$ and $s=t$ we get the fundamental equality

$$\mathbb{P}_\mu(\exp(-X_t(\phi)) = \exp(-\mu(V_t\phi)). \tag{9.47}$$

The extension from $\phi \geq 0$ in $D(A)$ to $\phi \geq 0$ in $b\mathcal{E}$ follows easily by considering the "weak form"

$$V_t\phi = P_t\phi - \frac{\gamma}{2}\int_0^t P_{t-s}(V_s\phi)^2 ds$$

and then taking bounded pointwise limits.

(9.47) proves the uniqueness of the law $\mathbb{P}_\mu(X_t \in \cdot)$ for any solution of (MP) and hence the uniqueness of $\mathbb{P}_\mu$ (see [571] or [226], Chapt. 4, Theorem 4.2).

(c) follows by taking $\phi \equiv 1$ and comparing the Laplace transforms of the transition measures.

■

Corollary 9.15 *(Infinite divisibility) $\mathbb{P}_\mu$ is an infinitely divisible probability measure with canonical representation*

$$(9.48)\quad \begin{aligned} &-\log(\mathbb{P}_\mu(\exp^{(-X_t(\phi))}) \\ &= -\int \log(\mathbb{P}_{\delta_x}(\exp^{(-X_t(\phi))})\mu(dx) = \int_{M_F(\mathbb{R}^d)\backslash\{0\}} (1-e^{\nu(\phi)})R_t(x,d\nu) \end{aligned}$$

where the canonical measure $R_t(x,d\nu) \in M_F(M_F(\mathbb{R}^d)\backslash\{0\})$ satisfies $R_t(x, M_F(\mathbb{R}^d)\backslash\{0\}) = \frac{2}{\gamma t}$ and the normalized measure is an exponential probability law with mean $\frac{\gamma t}{2}$.

The infinite divisibility of SBM allow us to use the the theory of infinitely divisible random measure (see e.g. Dawson (1992) (1993), [120], [140]) to obtain detailed properties of the process.

Weighted occupation time

If $\{X_t : t \geq 0\}$ is a super-Brownian motion, then we defined the associated *weighted occupation time* $Y_y : t \geq 0$ as

$$(9.49)\quad Y_t(A) = \int_0^t X_s(A)ds, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Theorem 9.16 *(Iscoe (1986) [330])*
Let $\mu \in M_F(\mathbb{R}^d)$ and $\phi, \psi \in C^2_{c,+}(\mathbb{R}^d)$. Then the joint Laplace functional of X_t and Y_t is given by

$$(9.50)\quad E_\mu\left[e^{-\langle\psi,X_t\rangle - \langle\phi,Y_t\rangle}\right] = e^{-\int u(t,x)\mu(dx)}$$

where $u(.,.)$ is the unique solution of

$$(9.51)\quad \begin{aligned} &\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) - \frac{\gamma}{2}u^2(t,x) + \phi(x), \\ &u(0,x) = \psi(x). \end{aligned}$$

Proof. The idea is to approximate the integral $\int_0^t X_s(\phi)ds$ by a Riemann sum approximation as follows:

$$E_\mu\left\{\exp\left[-\langle\psi,X_t\rangle-\int_0^t\langle\phi,X_s\rangle ds\right]\right\}=\lim_{N\to\infty}E_\mu\left\{\exp\left[-\langle\phi,X_t\rangle-\sum_{n=1}^N\langle\phi,X_{\frac{nt}{N}}\rangle\frac{t}{N}\right]\right\}$$

Now consider two semigroups on $C(\mathbb{R}^d)$:

- $V_t\psi$ given by the solution of

  $$(9.52)\quad v(t)=S_t\psi-\int_0^t S_{t-s}(v(s))^2ds,$$

- W_t given by

  $$(9.53)\quad W_t\psi=\psi+t\phi,\qquad \dot W_t\psi=\phi.$$

Using the iterated conditioning and the Markov property we obtain

$$(9.54)\quad E_\mu\left(\exp\left[-\langle\psi,X_y\rangle-\int_0^t\langle\phi,X_s\rangle ds\right]\right)$$
$$=\lim_{N\to\infty}\exp\left[-\langle(V_{\frac{t}{N}}W_{\frac{t}{N}})^N\psi,\mu\rangle\right]$$

Then by the Trotter-Lie product formula (cf. Chorin et al [74])

$$(9.55)\quad U_t=\lim_{N\to\infty}\left(V_{\frac{t}{N}}W_{\frac{t}{N}}\right)^N\qquad\text{on } C_{0,+}(\mathbb{R}^d).$$

and therefore

$$(9.56)\quad E_\mu\left(\exp\left[-\langle\psi,X_t\rangle-\int_0^t\langle\phi,X_s\rangle ds\right]\right)=\exp\left(-\langle U_t\psi_t,\mu\rangle\right)$$

noting that the interchange of limit and integral is justified by dominated convergence since

$$(9.57)\quad (V_{\frac{t}{N}}W_{\frac{t}{N}})^N\psi\le(S_{\frac{t}{N}}W_{\frac{t}{N}})^N\psi\le S_t\psi+(1+t)\|\phi\|.$$

■

As an application of this Iscoe established the *compact support property* of super-Brownian motion:

Theorem 9.17 *Let $\{X_t:t\ge 0\}$ with be super-Brownian motion with initial measure δ_0. Then*

$$(9.58)\quad P_{\delta_0}(\sup_{0\le t<\infty}X_t(\mathbb{R}^d\setminus\overline{B(0,R)})>0)=1-e^{-\frac{u(0)}{R^2}},$$

where $u(\cdot)$ is the solution of

$$\Delta u(x) = u^2(x), \quad x \in B(0,1), \qquad (9.59)$$
$$u(x) \to \infty \text{ as } x \to \partial B(0,1).$$

Proof. Iscoe (1988) [332]. ■

Convergence of renormalized BRW and interacting Feller CSBP

A number of different variations of rescaled branching systems on $\mathbb{R}^d$ can be proved to converge to SBM. For example, the following two results can be proved following the same basic steps.

Theorem 9.18 *Let $\varepsilon = \frac{1}{N}$. Consider a sequence of branching random walks X_t^ε on $\varepsilon\mathbb{Z}^d$ with random walk kernel $p^\varepsilon(\cdot)$ which satisfies (9.2), particle mass $m_\varepsilon = \varepsilon$, branching rate $\gamma^\varepsilon = \frac{\gamma}{\varepsilon}$ and assume that $X_0^\varepsilon \Rightarrow X_0$ in $\mathcal{M}_F(\mathbb{R}^d)$. Then $\{X_t^\varepsilon\}_{t\geq 0} \Rightarrow \{X_t\}_{t\geq 0}$ where X_t is a super-Brownian motion on $\mathbb{R}^d$ with $A = \frac{\sigma^2}{2}\Delta$ and branching rate γ.*

Remark 9.19 *The analogue of these results for more general branching mechanisms with possible infinite second moments are established in Dawson (1993), [140] Theorem 4.6.2.*

Theorem 9.20 *Consider a sequence of interacting Feller CSBP in which the rescaled random walks converge to Brownian motion. Then the interacting Feller CSBP converge to SBM.*

9.5.3 The Poisson Cluster Representation

Let X be an infinitely divisible random measure on a Polish space E with finite expected total mass $E[X(E)] < \infty$. Then (cf. [140], Theorem 3.3.1) there exists a measure $X_D \in M_F(E)$ and a measure $R \in M(M_F(E)\backslash\{0\})$ satisfying

$$\int (1 - e^{-\mu(E)})R(d\mu) < \infty$$

and such that

$$-\log(P(e^{-X(\phi)})) = X_D(\phi) + \int (1 - e^{-\nu(\phi)})R(d\nu).$$

X_D is called the deterministic component and R is called the *canonical measure.* For example, for a Poisson random measure with intensity Λ, $R(d\nu) = \int \delta_{\delta_x}(d\nu)\Lambda(dx)$. If we replace each Poisson point, x, with a random measure (*cluster*) with probability law $R(x, d\nu)$ then we obtain

$$R(d\nu) = \int \Lambda(dx)R(x, d\nu).$$

Exercise: Lectures 11-12.

Let $B_i(t), \;\; i = 1, \dots N$ be independent Brownian motions in $\mathbb{R}^d$ and define the measure-valued process

$$X_t^{(N)}(A) = \sum_{i=1}^{N} 1_A(B_i(t)), \;\; A \in \mathcal{B}(\mathbb{R}^d).$$

Assume that as $N \to \infty$, $X_0^{(N)}(A) \Longrightarrow |A \cap B(0,1)|$ where $|A|$ denotes the Lebesgue measure of A.

(a) Find the generator of the process $\{X_t\}$ acting on the class of functions of the form

$$F(\mu) = f(\mu(\phi)), \quad \phi \in C_b^2(\mathbb{R}^d), \; f \in C^2(\mathbb{R}).$$

(b) Prove the *dynamical law of large numbers:*

$$\{X_t^{(N)} : t \geq 0\} \Rightarrow \{X_t : t \geq 0\} \quad \text{as } N \to \infty$$

(in the sense of weak convergence on $C_{M_F(\mathbb{R}^d)}([0,\infty))$) where

$$X_t(A) = \int_A \int_{B(0,1)} p(t, x-y) dx dy.$$

where $p(t,x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t}$.