7.4 Dual martingale problems

Dual processes play an important role in the study of interacting particle systems (see Liggett [436]). A dual representation for the Fleming-Viot process was introduced in Dawson and Hochberg (1982) [131]. The following generalization with applications to measure-valued processes was established in (Dawson-Kurtz (1982) [143]). Here we give the main ideas and refer [140], Sect. 5.5 for the details.

To give the main idea we first present the theorem in a simplified case.

Theorem 7.9 *(Dual Representation)*

Let E_1, E_2 be Polish spaces and $F(\cdot,\cdot)$, $GF(\cdot,\cdot)$, $HF(\cdot,\cdot) \in \mathcal{B}_b(E_1 \times E_2)$, $\beta \in \mathcal{B}_b(E_2)$ and $P_x : E_1 \to \mathcal{P}(D_{E_1}(0,\infty))$ and $Q_y : E_2 \to \mathcal{P}(D_{E_2}(0,\infty))$

Assume that

$$(7.20)\qquad F(X(t),y) - \int_0^t GF(X(s),y)ds \text{ is a } P_{X(0)} \text{ martingale for each } y \in E_2$$

$$F(x,Y(t)) - \int_0^t HF(x,Y(s))ds \text{ is a } Q_{Y(0)} \text{ martingale for each } x \in E_1$$

and

$$(7.21)\quad GF(x,y) = HF(x,y) + \beta(y)F(x,y).$$

Then

$$(7.22)\quad E_x^X(F(X(t),y)) = E_y^Y\left(F(x,Y(t))\exp(\int_0^t \beta(Y(s))ds\right), \qquad 0<t<T$$

Proof. Let

$$(7.23)\quad \Phi(s,t) := E_x^X \otimes E_y^Y\left(F(X(s),Y(t))\exp(\int_0^t \beta(Y(u))du)\right)$$

$$(7.24)\quad \Phi(t,0) = E_x^X\left(F(X(t),y)\right)$$

$$(7.25)\quad \Phi(0,t) = E_y^Y\left(F(x,Y(t))\right)$$

$$(7.26)\quad \Phi_1(s,t) = E_x^X \otimes E_y^Y\left(GF(X(s),Y(t))\exp(\int_0^t \beta(Y(u))du)\right)$$

(7.27)

$$\Phi_2(t,s) = E_x^X \otimes E_y^Y\left([HF(X(t),Y(s)) + \beta Y(s)F(X(t),Y(s))]\exp(\int_0^s \beta(Y(u))du)\right)$$

where Φ_1, Φ_2 are the first partial derivatives with respect to the first and second variables. Under the assumptions, $\Phi_1(s,t-s), \Phi_2(s,t-s),\ 0 \le s \le t$ exist and are uniformly bounded .

Therefore

$$\Phi(0,t) - \Phi(t,0) = \int_0^t \frac{\partial}{\partial s}\Phi(s,t-s) = \int_0^t (\Phi_1(s,t-s) - \Phi_2(s,t-s))ds = 0 \tag{7.28}$$

■

In applications the assumption that $\beta(\cdot)$ and $GF(\cdot,\cdot)$ are bounded needs to be relaxed. The following extension (see [140], Cor. 5.5.3) provides the required conditions.

Proposition 7.10 *Assume that*

(i) $F \in C_b(E_1 \times E_2)$, *and* $\{F(\cdot,y) : y \in E_2)$ *is measure-dtermining on* E_1

(ii) there exist stopping times $\tau_K \uparrow t$ *such that*

$$\left\{(1 + \sup_x |GF(x, Y(\tau_K))|) \cdot \exp(\int_0^{\tau_K} |\beta(Y(u))|du)\right\}_K \tag{7.29}$$

are Q_{δ_y} *− uniformly integrable for all* $y \in E_2$

and (iii) $Q_{\delta_y}(Y(s-) \neq Y(s)) = 0$ *for each* $s \ge 0$*, that is, no fixed discontinuities.*

Then the G-martingale problem is well-posed and for all $y \in E_2$

$$P_\mu(F(X(t),y)) = \int_{E_1} \mu(dx)\left(Q_{\delta_y}[F(x,Y(t))\exp\left(\int_0^t \beta(Y(u))du\right)\right). \tag{7.30}$$

Example 7.11 *(The Wright-Fisher diffusion with polynomial drift)*

Let $\Delta_{d-1} = \{(x_1,\ldots,x_d), x_i \ge 0, i = 1,\ldots,d,\ \sum_{i=1}^d x_i \le 1\}$

Then consider the Wright-Fisher diffusion $\{x(t)\}$ *with generator*

$$G = \sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}$$

where $\{a_{i,j}(x)\}$ *is the real symmetric non-negative definite matrix,* $\{a_{ij}(x)\} = \{x_i(\delta_{ij} - x_j)\}$ *and the drift coefficient* $b_i(x)$ *is a polynomial satisfying certain natural boundary conditions on* Δ_{d-1} *to ensure that the process remains in* Δ_{d-1}.

Shiga (1981)) [541] obtained a dual in terms of a family of functions $\{\phi_\alpha\}_{\alpha\in\Gamma}$, $\phi_\alpha \in D(G)$ *defined by*

$$\phi_\alpha(x_1,\ldots,x_d) = \prod_{i=1}^d x_i^{\alpha_i}, \qquad \alpha = (\alpha_1,\ldots,\alpha_d) \in \Gamma$$

and showed that

$$G\phi_\alpha = \sum_\beta Q_{\alpha,\beta}(\phi_\beta - \phi_\alpha) + h_\alpha \phi_\alpha$$

where $Q = \{Q_{\alpha,\beta}\}$ *defines a conservative Markov chain* α_t *with state space* Γ. *Then the following identity follows from Proposition 7.10:*

$$(7.31)\quad E_x[\phi_\alpha(x(t))] = E_\alpha[\phi_{\alpha_t}(x)\exp(\int_0^t h_{\alpha_u} du)],\ 0 \le t \le t_0$$

provided that

$$E_\alpha[\exp(\int_0^{t_0} |h_{\alpha_u}|du)] < \infty \quad \forall \alpha \in \Gamma$$

Therefore the corresponding Wright-Fisher martingale problem is well-posed.

Example 7.12 *(Markov chains) Consider a continuous time Markov chain with state space* $E_K = \{1, \ldots, K\}$ *and transition rates*

$$(7.32)\quad i \to j \text{ with rate } m_{ij}$$

Let $\mathcal{PK}$ *denote the collection of subsets of* E_K *and define the function* $F : \mathcal{PK} \times E_K$ *by*

$$(7.33)\quad F(A, j) = 1_A(j)$$

.

Now consider the Markov $\mathcal{A}_t$ *chain with state space* $\mathcal{PK}$ *and transition rates*

$$(7.34)\quad A \to A \cup \{j\} \text{ at rate } \sum_{\ell \in A} m_{j\ell}$$

$$(7.35)\quad A \to A\backslash\{j\} \text{ at rate } \sum_{\ell \in A^c} m_{j\ell} \quad \text{if } j \in A$$

(7.36)

$$GF(A,j) = \sum_\ell m_{j\ell}(1_A(\ell) - 1_A(j)) = \sum_{\ell \in A} m_{j\ell}(1 - F(A,j)) - \sum_{\ell \in A^c} m_{j\ell} F(A,j)$$

Then

(7.37)

$$HF(A) = \sum_{k \in A^c}[(\sum_{\ell \in A} m_{k\ell})(F(A \cup \{k\}) - F(A)] + \sum_{k \in A}[(\sum_{\ell \in A^c} m_{k\ell})(F(A\backslash\{k\}) - F(A))]$$

and therefore

$$HF(A,j) = \sum_{k\in A^c}(\sum_{\ell\in A} m_{k\ell}))(1_{(A\cup\{k\})}(j)-1_A(j))+\sum_{k\in A}(\sum_{\ell\in A^c} m_{k\ell})(1_{(A\backslash\{k\})}(j)-1_A(j)) \tag{7.38}$$

$$= \sum_{\ell\in A} m_{j\ell}(1-F(A,j))-\sum_{\ell\in A^c} m_{j\ell}F(A,j).$$

By duality we have

$$E_j(1_\ell(X_t)) = E_{\{\ell\}}(1_{\mathcal{A}_t}(j)). \tag{7.39}$$

Remark 7.13 *If $\{m_{ij}\}$ is irreducible, then the Markov chain $\mathcal{A}_t$ has two traps $\emptyset$ and E_K. It is easy to verify that $\mathcal{A}_t$ is absorbed at a trap with probability one. This together with (7.39) implies (the elementary result) that $P_j(x(t)=\ell)$ converges as $t\to\infty$ to a stationary measure π_ℓ with*

$$\pi_\ell = P_{\{\ell\}}(\mathcal{A}_t\to E_K), \quad \ell\in E_K \tag{7.40}$$

and that $\lim_{t\to\infty} P_j(X(t)=\ell)$ *is independent of* j.

7.5 Dual representation of the neutral Fleming-Viot process

The method of dual representation plays an important role in the study of Fleming-Viot processes and will be frequently used below. To introduce this we first consider the special case of a neutral Fleming-Viot process with a nice mutation process.

7.5.1 The General Neutral F.V. Process

Let E be a compact metric space, A be a linear operator defined on $D(A)\subset C(E)$ and assume that the closure of A generates a Feller semigroup, $\{S_t : t\geq 0\}$ on $C(E)$. A probability measure $\mathbb{P}_\mu$ on $C([0,\infty),M_f(E))$ is said to be a solution of the *neutral Fleming-Viot martingale problem* $\mathbb{MP}_{(A,Q,0)}$ with initial condition μ if

$$\mathbb{P}_\mu(X_0=\mu)=1$$

and for each $\phi\in C_b^+(E)\cap D(A)$

$$M_t^0(\phi) := \langle\phi,X_t\rangle-\langle\phi,X_0\rangle-\int_0^t\langle A\phi,X_s\rangle\,ds$$

where M_t^0 defines a martingale measure $M^0(ds,dx)$ with covariance

$$\left\langle M^0(dx),M^0(dy)\right\rangle_t = \gamma\int_0^t Q(X_s;dx,dy)ds$$

$$Q(\mu;dx,dy) = \delta_x(dy)\mu(dx)-\mu(dx)\mu(dy).$$

Theorem 7.14 *There exists a unique solution to the* $\mathbb{MP}_{(A,Q,0)}$ *martingale problem.*

Proof. This will be proved in the following section. ■

7.5.2 Equivalent Formulation of the martingale problem

We now turn to an equivalent formulation of the Fleming-Viot process that will be needed for the application of the dual representation in the next chapter.

Let $F \in D(G) \subset\in C^2(\mathcal{P}(E))$

$$(7.41)\quad GF(\mu) = \int_E \left(A\frac{\delta F(\mu)}{\delta\mu(x)} \right) \mu(dx) + \frac{\gamma}{2}\int_E\int_E \frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)} Q(\mu; dx, dy)$$

where $Q(\mu, dx, dy) := \mu(dx)\delta_x(dy) - \mu(dx)\mu(dy)$.

Now consider function $F(\mu,(f,n)) = \int \dots \int f(x_1,\dots,x_n)\mu^n(dx)$ with $f \in C(E^n)$, $n \in \mathbb{N}$ and

$$(7.42)\quad \mu^n(dx) = \mu(dx_1)\dots\mu(dx_n).$$

Then

$$(7.43)\quad GF(\mu,(f,n)) = \langle \mu^n, A^{(n)} f\rangle + \frac{\gamma}{2}\sum_{i\neq j}\Big(\langle \mu^{n-1}, \widetilde{\Theta}_{ij} f\rangle - \langle \mu^n, f\rangle\Big)$$

$$(7.44)\quad (\widetilde{\Theta}_{ij} f)(y_1,\dots,y_{N-1}) := f(x_1,\dots,x_N)$$

On the right side of (7.44)

$$(7.45)\quad \begin{aligned} &x_k = y_k \text{ for } k < i\vee j,\ k \neq i\wedge j \\ &x_{i\vee j} = x_{i\wedge j} = y_{i\wedge j} \\ &x_k = y_{k-1} \text{ for } k > i \vee j. \end{aligned}$$

We will interpret this below as the *coalescence* of dual particles. The dual particle system of coalescing Markov processes leads to the *Kingman coalescent* -see Section 8.2.5 for details.

7.5.3 The dual process

The Fleming-Viot process has state space $\mathcal{P}(E)$. We assume that the mutation process has semigroup S_t with generator A and there exists a dense set $D_0 \subset C(E)$ and $S_t : D_0 \to D_0$.

We can then consider the extension of the mutation process to E^n, $n \geq 1$ corresponding to n i.i.d. copies of the basic mutation process and with generator $A^{(n)} = \sum_{i=1}^n A_i$ where A_i denotes the action of A on the ith variable.

Consider the algebra of functions, $\mathcal{C}(E^{\mathbb{N}})$, of the form

$$(7.46)\quad f=\sum_{n=1}^{\infty} f_n, \qquad f_n\in C(E^n), \qquad f_n=0\ a.a.\,n$$

$$(7.47)\quad \mathfrak{n}(f)=\max\{m: f_m\neq 0\}.$$

Define $F:\mathcal{C}(E^{\mathbb{N}})\times\mathcal{P}(E)$ by

$$(7.48)\quad F(f,\mu)=\sum_{n=1}^{\mathfrak{n}(f)}\int_{E^n} f_n(x_1,\dots,x_n)d\mu^{\otimes n}$$

A function $f\in\mathcal{C}(E^{\mathbb{N}})$ is said to be simple if $f_m=0$ for all $m<\mathfrak{n}(f)$. The set of simple functions is denoted by $\mathcal{C}_{\text{sim}}(E^{\mathbb{N}})$.

Now consider the Fleming-Viot process with MP generator: for each $f\in D_0(A)$

$$(7.49)\quad GF(f,\mu)=\int_E\left(A\frac{\partial F(f,\mu)}{\partial\mu(x)}\right)\mu(dx)+\frac{\gamma}{2}\int_E\int_E\frac{\partial^2 F(f,\mu)}{\partial\mu(x)\partial\mu(y)}Q(\mu;dx,dy)$$

and note that for for each $\mu\in\mathcal{P}(E)$ this coincides with

$$(7.50)\quad KF(f,\mu)=F(Af,\mu)+\frac{\gamma}{2}\sum_{j=1}^{\mathfrak{n}(f)}\sum_{k\neq j}[F((\widetilde{\Theta}_{jk}f,\mu)-F(f,\mu))]$$

where

$$(7.51)\quad Af=\sum_m A^{(m)}f_m.$$

where $\widetilde{\Theta}_{jk}:D_0^n\to D_0^{n-1}$ is defined by (7.44).

Then K is the generator of a càdlàg process with values in $\mathcal{C}_{\text{sim}}(E^{\mathbb{N}})$ and law $\{Q_f: f\in\mathcal{C}_{\text{sim}}(E^{\mathbb{N}})\}$ which evolves as follows:

- $Y(t)$ jumps from $C(E^n)$ to $C(E^{n-1})$ at rate $\frac{1}{2}\gamma n(n-1)$
- at the time of a jump, f is replaced by $\widetilde{\Theta}_{jk}f$
- between jumps, $Y(t)$ is deterministic on $C(E^n)$ and evolves according to the semigroup (S_t^n) with generator $A^{(n)}$.

Theorem 7.15 *(a) Let* $(\{X(t)\}_{t\geq 0},\{P_\mu:\mu\in\mathcal{P}(E)\})$ *be a solution to the Fleming-Viot martingale problem and the process* $(\{Y(t)\}_{t\geq 0},\{Q_f: f\in C_{\text{sim}}(E^{\mathbb{N}})\})$ *be defined as above. Then*