Problem 1. Let \( \{a_n\}_{n \in \mathbb{N}} \) be the sequence given by
\[
a_1 = 1 \quad \text{and} \quad a_{n+1} = 3a_n + \left\lfloor \sqrt{5} \cdot a_n \right\rfloor \quad \text{for} \ n \geq 1.
\]
Compute \( a_{2021} \).

Problem 2. Let \( n \in \mathbb{N} \). Find the number of pairs of polynomials \((P(x), Q(x)) \in \mathbb{R}[x] \times \mathbb{R}[x]\) satisfying the following two conditions:
- \( \deg(P) > \deg(Q) \); and
- \( P^2(x) + Q^2(x) = x^{2n} + 1 \).

Problem 3. Let \( k \in \mathbb{N} \). Prove that there exist polynomials \( P_0, P_1, \ldots, P_{k-1} \) (which may depend on \( k \)) with the property that for each \( n \in \mathbb{N} \), we have
\[
\left\lfloor \frac{n}{k} \right\rfloor = P_0(n) + P_1(n) \cdot \left\lfloor \frac{n}{k} \right\rfloor + P_2(n) \cdot \left\lfloor \frac{n}{k} \right\rfloor^2 + \cdots + P_{k-1}(n) \cdot \left\lfloor \frac{n}{k} \right\rfloor^{k-1},
\]
where (as always) \( \lfloor x \rfloor \) is the integer part of the real number \( x \).

Problem 4. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) with the property that
\[
f(x, y) + f(y, z) + f(z, x) = 0,
\]
for all real numbers \( x, y \) and \( z \). Prove that there must exist another function \( g : \mathbb{R} \to \mathbb{R} \) such that
\[
f(x, y) = g(x) - g(y),
\]
for all real numbers \( x \) and \( y \).