Problem 1. Find a polynomial \( P(x,y) \in \mathbb{R}[x,y] \) with the property that for each real number \( r \), we have
\[
P([r],[2r]) = 0,
\]
where \([x]\) is always the integer part of the real number \( x \) (i.e., the largest integer less than or equal to \( x \)).

Solution. We let
\[
P(x,y) = (y - 2x)(y - 2x - 1)
\]
and note that for each real number \( r \), we have that
\[
either [2r] = 2 \cdot [r], or [2r] = 2[r] + 1,
\]
which means that \( P([r],[2r]) = 0 \) for each \( r \in \mathbb{R} \).

Problem 2. Show that the curve in the cartesian plane given by the equation:
\[
x^3 + 3xy + y^3 = 1
\]
contains exactly one set of three points \( A, B \) and \( C \) which are the vertices of an equilateral triangle.

Solution. The whole key to this problem is the following factorization:
\[
x^3 + y^3 + 3xy - 1 = (x + y - 1)(x^2 + y^2 + 1 - xy + x + y)
\]
which comes from the identity:
\[
x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).
\]
Now, using the fact that
\[
x^2 + y^2 + 1 - xy + x + y = \frac{1}{2} \cdot (x - y)^2 + \frac{1}{2} \cdot (x + 1)^2 + \frac{1}{2} \cdot (y + 1)^2,
\]
we get that besides the line \( x + y = 1 \), the given plane curve contains only the point \((-1,-1)\). So, indeed, there is only one triple of points on the given curve which are the vertices of an equilateral triangle; one of those three points must be \((-1,-1)\), while the other two points lie on the line \( x + y = 1 \) being exactly \( \frac{\sqrt{3}}{2} \cdot h \) units apart from the point \((\frac{1}{2},\frac{1}{2})\), which is the foot of the perpendicular line from \((-1,-1)\) to the line \( x + y = 1 \), where \( h \) is the length of the height from \((-1,-1)\) to this line, i.e.,
\[
h = \sqrt{2} \cdot \frac{3}{2}.
\]
So, the other two vertices of the equilateral triangle are
\[
\left(\frac{1 + \sqrt{3}}{2}, \frac{1 - \sqrt{3}}{2}\right) \quad \text{and} \quad \left(\frac{1 - \sqrt{3}}{2}, \frac{1 + \sqrt{3}}{2}\right).
\]
Problem 3. Let \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence of integers satisfying the two properties:

\[
a_i = i \text{ for } i = 1, \ldots, 2020 \quad \text{and} \quad a_n = a_{n-1} + a_{n-2020} \text{ for } n \geq 2021.
\]

Show that for each positive integer \( M \), there exists some integer \( k > M + 2020 \) such that each one of the integers \( a_k, \ldots, a_{k+2018} \) are divisible by \( M \).

Solution. We extend the definition of the sequence \( \{a_n\} \) for all \( n \in \mathbb{Z} \) simply by enforcing the condition

\[
a_n = a_{n-1} + a_{n-2020}
\]

for all \( n \in \mathbb{Z} \). Note that we can solve for \( a_0 \) from

\[
a_{2020} = a_{2019} + a_0
\]

and get \( a_0 = 1 \). Similarly, we solve for \( a_{-1} \) from

\[
a_{2019} = a_{2018} + a_{-1}
\]

and get \( a_{-1} = 1 \). Furthermore, \( a_{-k} = 1 \) for each \( k \in \{0, 1, \ldots, 2018\} \). Then we have \( a_{-2019} = 0 \) because

\[
a_1 = a_0 + a_{-2019}
\]

and \( a_1 = a_0 = 1 \). Continuing to solve backwards, we get

\[
a_{-k} = 0 \text{ for } k = 2019, 2020, \ldots, 4037.
\]

For example, note that

\[
a_{-2017} = a_{-2018} + a_{-4037}
\]

and \( a_{-2017} = a_{-2018} = 1 \).

Therefore, there exist 2019 consecutive integers in our recurrence sequence, all of them equal to 0.

On the other hand, for any given positive integer \( M \), any recurrence sequence is eventually periodic modulo \( M \). Furthermore, since for our sequence we can solve also backwards (as shown above), the sequence is actually periodic modulo \( M \). (The same trick can be applied to the Fibonacci sequence, for example, to show that for any integer \( M \) there exist infinitely many terms in the Fibonacci sequence all of them divisible by \( M \).)

So, since at one point we had 2019 consecutive integers in our sequence all divisible by \( M \) (simply because those integers are all equal to 0), then we can find such consecutive integers divisible by \( M \) in our sequence with indices arbitrarily large.

Just to give more details to our reasoning: first of all, since there exist finitely many residue classes modulo \( M \) (for any given positive integer \( M \)), there must exist two distinct tuples of 2020 consecutive elements in our sequence which give us the same residue classes modulo \( M \). So, there exist two distinct 2020 consecutive tuples of elements in our sequence

\[
(a_k, a_{k+1}, \ldots, a_{k+2019}) \text{ and } (a_\ell, a_{\ell+1}, \ldots, a_{\ell+2019})
\]

such that \( a_{k+i} \equiv a_{\ell+i} \pmod{M} \) for each \( i = 0, 1, \ldots, 2019 \), then our linear recurrence formula yields that

\[
a_{k+2020} \equiv a_{k+2019} + a_k \equiv a_{\ell+2019} + a_\ell \equiv a_{\ell+2020} \pmod{M}
\]

and more generally, inductively, we get that for each nonnegative integer \( i \), we have that

\[
a_{k+i} \equiv a_{\ell+i} \pmod{M}.
\]
But also, going backwards, we have
\[ a_{k-1} \equiv a_{k+2019} - a_{k+2018} \equiv a_{\ell+2019} - a_{\ell+2018} \equiv a_{\ell-1} \pmod{M} \]
and then also, for all \( i \in \mathbb{N} \), we have
\[ a_{k-i} \equiv a_{\ell-i} \pmod{M}, \]
thus showing that our linear recurrence sequence is periodic modulo \( M \). Since at one moment (for the indices \( k = -2019, -2020, \ldots, -4037 \) we have 2019 consecutive integers in our sequence all divisible by \( M \) (since in that case, they’re all equal to 0), then the same phenomenon repeats infinitely often, i.e., there exist arbitrarily large positive integers \( k \) such that \( a_k, a_{k+1}, \ldots, a_{k+2018} \) are all divisible by \( M \), as desired.

**Problem 4.** Let \( n \) be a positive integer and let \( \theta \in \mathbb{R} \) such that \( \theta/\pi \) is an irrational number. For each \( k = 1, \ldots, n \), we let
\[ a_k = \tan \left( \theta + \frac{k\pi}{n} \right). \]
Compute \( \frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n} \).

**Solution.** We let \( \omega := e^{2\theta n \cdot i} = \cos(2n\theta) + i \sin(2n\theta) \).

For the polynomial
\[ P(x) = (1 + ix)^n - \omega \cdot (1 - ix)^n, \]
we compute for each \( k = 1, \ldots, n \) that
\[ P(a_k) = \left( \frac{\cos \left( \theta + \frac{k\pi}{n} \right) + i \sin \left( \theta + \frac{k\pi}{n} \right)}{\cos \left( \theta + \frac{k\pi}{n} \right)} \right)^n - \omega \cdot \left( \frac{\cos \left( \theta + \frac{k\pi}{n} \right) - i \sin \left( \theta + \frac{k\pi}{n} \right)}{\cos \left( \theta + \frac{k\pi}{n} \right)} \right)^n \]
and so, letting
\[ \varepsilon_k := e^{(n\theta + k\pi) - i}, \]
we see that
\[ P(a_k) = \frac{\varepsilon_k - \omega \cdot \varepsilon_k}{\cos^n \left( \theta + \frac{k\pi}{n} \right)} = 0 \]
because
\[ \frac{\varepsilon_k}{\varepsilon_k} = e^{2n\theta \cdot i} = \omega. \]
In conclusion, the polynomial \( P(z) \) vanishes at each point \( a_k \) for \( k = 1, \ldots, n \) and since it also has degree \( n \) and leading coefficient equal to
\[ e_n := i^n - \omega \cdot (-i)^n, \]
we conclude that
\[ P(z) = c_n \cdot \prod_{k=1}^{n} (z - a_k). \]
So,
\[ \frac{\sum_{k=1}^{n} a_k}{\prod_{k=1}^{n} a_k} = \frac{-e_{n-1}}{(-1)^n e_0} \]
where we write
\[ P(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0. \]
Clearly,

\[ c_0 = 1 - \omega \quad \text{and} \quad c_{n-1} = n\omega^{n-1} - \omega \cdot n(-1)^{n-1} = n^{n-1} \cdot (1 + \omega(-1)^n), \]

which means that

\[ \sum_{k=1}^{n} a_k \prod_{k=1}^{n} a_k = \frac{1 + \omega(-1)^n}{1 - \omega} \cdot n(-i)^{n-1}. \]

As a fun fact, if \( n \) is odd, then the above quotient is always an integer because then \( 1 + \omega(-1)^n = 1 - \omega \).