PUTNAM PRACTICE SET 25: SOLUTIONS

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Problem 1. Let \( n \in \mathbb{N} \) and let \( a_1, \ldots, a_n \in \mathbb{R} \). Show that there exists an integer \( m \) and some nonempty subset \( S \subseteq \{1, \ldots, n\} \) with the property that

\[
\left| m + \sum_{i \in S} a_i \right| \leq \frac{1}{n+1}.
\]

Solution. We consider the fractional parts \( \{ \cdot \} \) of the following numbers:

\[
s_k := \sum_{i=1}^{k} a_i \quad \text{for} \quad k = 1, \ldots, n.
\]

Case 1. There exists \( 1 \leq i < j \leq n \) such that

\[
|\{s_j\} - \{s_i\}| \leq \frac{1}{n+1}.
\]

In this case, writing \( s_j = \{s_j\} + m_j \) and \( s_i = \{s_i\} + m_i \) for some integers \( m_i \) and \( m_j \) (actually their respective integer parts \([\cdot]\)), then we get:

\[
|s_j - m_j - (s_i - m_i)| \leq \frac{1}{n+1},
\]

which means that

\[
\left| \sum_{i<k<j} a_k - (m_j - m_i) \right| \leq \frac{1}{n+1}.
\]

So, letting \( m := m_i - m_j \), then we obtain the desired conclusion.

Case 2. For each \( i \neq j \), we have that

\[
|\{s_j\} - \{s_i\}| > \frac{1}{n+1}.
\]

In this case, ordering the \( n \) fractional parts \( \{s_k\} \) for \( 1 \leq k \leq n \), we see that they live in \([0,1)\) and the distance between any two of them is greater than \( \frac{1}{n+1} \), which means that:

- either \( \{s_{i_0}\} \leq \frac{1}{n+1} \), where \( \{s_{i_0}\} \) is the smallest of the above fractional parts, in which case, the conclusion follows easily (we simply take \( S = \{1, \ldots, i_0\} \) and \( m = -\lfloor s_{i_0} \rfloor \)).
- or \( 1 - \{s_{j_0}\} < \frac{1}{n+1} \), where \( \{s_{j_0}\} \) is the largest of the above fractional parts. In this case, we take \( S = \{1, \ldots, j_0\} \) and \( m = -1 - \lfloor s_{j_0} \rfloor \) and still obtain the desired conclusion.

Problem 2. For each continuous function \( f : [0,1] \rightarrow \mathbb{R} \), let

\[
I(f) := \int_0^1 x^2f(x)dx - \int_0^1 xf(x)^2dx.
\]
Find the maximum of $I(f)$ over all possible continuous functions $f$.

**Solution.** We compute
\[ I(f) = \int_0^1 (x^2f(x) - xf^2(x)) \, dx = \int_0^1 x(xf(x) - f^2(x)) \, dx = \int_0^1 x \left( -\frac{x^2}{4} + xf(x) - f^2(x) \right) \, dx \]
and since
\[ \frac{x^2}{4} - xf(x) + f^2(x) = \left( \frac{x}{2} - f(x) \right)^2 \geq 0, \]
we see that
\[ I(f) \leq \int_0^1 x^3 \, dx = \frac{1}{16}. \]
The maximum $\frac{1}{16}$ is attained when $f(x) = \frac{x}{2}$ (which is a **continuous** function).

**Problem 3.** Let $c$ be a real number greater than 1 and let $g \in \mathbb{R}[x]$ be a non-constant polynomial with the property that there exists an infinite sequence $\{k_n\} \subseteq \mathbb{N}$ with the property that for each $n \geq 1$, we have that there exists some $\ell_n \in \mathbb{N}$ with the property that
\[ g(c^{k_n}) = c^{\ell_n}. \]
Find all such polynomials $g$.

**Solution.** Let $d \geq 1$ be the degree of the polynomial $g(x)$ and also, let $A$ be the leading coefficient of $g$. We consider the following limit:
\[ L := \lim_{n \to \infty} \frac{g(c^{\ell_n})}{c^{d\ell_n}}. \]
From basic calculus, it’s clear that $L = A$ since $c^{k_n} \to \infty$ as $n \to \infty$ (note that $c > 1$). On the other hand, we have that
\[ L = \lim_{n \to \infty} c^{\ell_n - dk_n} \]
and so, the limit $L$ exists and is non-zero if and only if there exists some integer $b$ such that for all $n$ sufficiently large, we have that
\[ \ell_n - dk_n = b \]
(note that $c > 1$ and so, powers of $c$ won’t accumulate near a nonzero real number). Hence $A = c^b$, but moreover, using also (1), we have that for each $x_n := c^{k_n}$, where $n$ is sufficiently large,
\[ g(x_n) = Ax_n^d. \]
So, the polynomial $h(x) := g(x) - Ax^d$ vanishes at each point $x_n$ (for $n$ sufficiently large) thus showing that $h$ must be identically equal to 0 (again note that the points $x_n$ are distinct because $c > 1$). So, always we have that
\[ g(x) = c^b \cdot x^d \]
for some $b \in \mathbb{Z}$.

**Problem 4.** Let $f : [0, 1] \to \mathbb{R}$ be a function whose derivative is continuous, which also satisfies $\int_0^1 f(x) \, dx = 0$. Prove that for each $\alpha \in (0, 1)$ we have
\[ \left| \int_0^\alpha f(x) \, dx \right| \leq \frac{1}{8} \cdot \max_{0 \leq x \leq 1} |f'(x)|. \]
Solution. We define the function \( g : [0, 1] \rightarrow \mathbb{R} \) given by
\[
g(x) := \int_0^x f(y)dy.
\]
Then \( g(0) = g(1) = 0 \) and clearly, \( g(x) \) is a function whose derivative (which is \( f(x) \)) is continuous. So, there exists a point - call it \( \alpha \) - inside the interval \( (0, 1) \) with the property that
\[
\left| \int_0^{\alpha} f(x)dx \right| \text{ is the largest.}
\]
Then \( x = \alpha \) is a critical point for the function \( g \) and thus,
\[
0 = g'(\alpha) = f(\alpha).
\]
So, since the maximum is attained at \( x = \alpha \), it suffices to prove that
\[
\left| \int_0^{\alpha} f(x)dx \right| \leq \frac{M}{8},
\]
where \( M := \max_{0 \leq x \leq 1} |f'(x)| \).

We may assume that \( \alpha \leq \frac{1}{2} \) since otherwise we may replace \( f(x) \) by \( f(1 - x) \) which leaves our hypotheses unchanged, while \( M \) would still be unchanged and also,
\[
\max_{0 \leq y \leq 1} \left| \int_0^{y} f(x)dx \right|
\]
would be unchanged, but this time \( \alpha \) would be replaced by \( 1 - \alpha \). So, from now on, we assume \( \alpha \leq \frac{1}{2} \).

Without loss of generality, we may assume that
\[
\int_0^{\alpha} f(x)dx > 0
\]
since otherwise we could just replace \( f(x) \) by \( -f(x) \) and still prove the same conclusion.

Now, because \( f(\alpha) = 0 \) and \( f'(x) \geq -M \), we conclude that
\[
f(x) \leq M(\alpha - x) \text{ for each } 0 \leq x \leq \alpha.
\]
So, since we also argued that we may assume that \( \alpha \leq \frac{1}{2} \), then we have:
\[
\left| \int_0^{\alpha} f(x)dx \right| = \int_0^{\alpha} f(x)dx \leq \int_0^{\alpha} M(\alpha - x)dx = \frac{Ma^2}{2} \leq \frac{M}{8}.
\]