EXISTENCE OF PENCILS WITH NONBLOCKING HYPERSURFACES

SHAMIL ASGARLI, DRAGOS GHIOCA, AND CHI HOI YIP

ABSTRACT. We prove that there is a pencil of hypersurfaces in $\mathbb{P}^n$ of any given degree over a finite field $\mathbb{F}_q$ such that every $\mathbb{F}_q$-member of the pencil is not blocking with respect to $\mathbb{F}_q$-lines.

1. INTRODUCTION

Let $\mathcal{P}$ represent some property a given algebraic hypersurface $X \subset \mathbb{P}^n$ may satisfy. For instance, the property $\mathcal{P}$ could be "is smooth", "is irreducible", or "has no rational points". There are multiple perspectives in which a given property $\mathcal{P}$ may hold for a generic hypersurface. When the base field is a finite field $\mathbb{F}_q$, which will be the case for our consideration, there are at least three natural ways to express how a given property $\mathcal{P}$ may be generic:

(1) (d is fixed) The natural density of hypersurfaces of degree d over $\mathbb{F}_q$ which satisfies $\mathcal{P}$ tends to 1 as $q \to \infty$, or at least, tends to some function $f(d)$ which is $1 - o_d(1)$ as $d \to \infty$.
(2) (q is fixed) The natural density of hypersurfaces of degree d which satisfies $\mathcal{P}$ tends to 1 as $d \to \infty$, or at least, tends to some function $f(q)$ which is $1 - o_q(1)$ as $q \to \infty$.
(3) (q, d are both fixed) The parameter space of hypersurfaces of degree d has large-dimensional linear spaces whose $\mathbb{F}_q$-points all correspond to hypersurfaces which satisfy $\mathcal{P}$.

The statements (1) and (2) can be viewed as saying that hypersurfaces satisfying $\mathcal{P}$ are abundant on a "global" level. In contrast, the statement (3) is about the "local" distribution of hypersurfaces satisfying $\mathcal{P}$. When q and d are sufficiently large, the statements (1) and (2) suggest, but do not directly imply the statement (3).

When the property $\mathcal{P}$ stands for smoothness, Lang-Weil bounds [LW54] imply that (1) holds, and Poonen's theorem [Poo04] computes the exact density of smooth hypersurfaces over $\mathbb{F}_q$ and justifies (2). For condition (3), the first two authors gave a positive answer to the existence of pencil of smooth hypersurfaces in [AG23] for sufficiently large q when d, n are fixed; the existence of large-dimensional families of smooth hypersurfaces was essentially settled in a subsequent work [AGR22] joint with Reichstein.

In this paper, we will investigate the condition when $\mathcal{P}$ represents the property that a hypersurface is nonblocking. We say that a hypersurface $X \subset \mathbb{P}^n$ is *nonblocking with respect to lines* if there exists an $\mathbb{F}_q$-line L such that $X \cap L$ has no $\mathbb{F}_q$-points. We will drop the part "with respect to lines" for brevity, and call such a hypersurface *nonblocking*. When d is fixed, and $q \to \infty$, almost all curves are smooth, hence irreducible, by the Lang-Weil bounds. Thus, almost all curves are nonblocking by [AGY22a, Theorem 1.2], settling (1) in dimension two. In our previous paper [AGY22b] we showed that most plane curves are nonblocking from an arithmetic statistics point of view, thus settling (2) in dimension two. The goal of the present paper is to illustrate the abundance of nonblocking hypersurfaces over finite fields by examining (3).

2020 *Mathematics Subject Classification.* Primary: 14N05, 14C21; Secondary: 14J70, 14G15, 51E21.
Key words and phrases. Pencil of hypersurfaces, blocking set, finite field.

The concept of blocking plane curves was formally introduced in [AGY22a] with a view towards the rich interplay between finite geometry and algebraic geometry. One of the main tools in the study of blocking sets is to consider an associated algebraic curve (or a variety in general) and study its geometry [SS98]. Our work focuses on the other direction, namely understanding when the points on a given algebraic variety forms a blocking set. Recall that a set of points $B \subseteq \mathbb{P}^n(\mathbb{F}_q)$ is a *blocking set* (with respect to lines) if every $\mathbb{F}_q$-line meets B. A blocking set B is *trivial* if it contains all the $\mathbb{F}_q$-points of a hyperplane defined over $\mathbb{F}_q$, and is otherwise said to be *nontrivial*. One question of particular interest is to determine the minimum size of a nontrivial blocking set; we refer to [BSS14] for a recent survey on related topics.

Our main theorem asserts the existence of completely nonblocking pencils. This precisely corresponds to the case of 1-dimensional (projective) linear subspaces in the statement (3) where the property $\mathcal{P}$ stands for "is nonblocking".

Theorem 1.1. *Let $n \geq 2$, $d \geq 2$, and q be an arbitrary prime power. There exists a pencil $\mathcal{L}$ of degree d hypersurfaces in $\mathbb{P}^n$ such that every $\mathbb{F}_q$-member of $\mathcal{L}$ is nonblocking.*

We remark that linear systems of hypersurfaces over finite fields have been investigated from a few different perspectives in the literature (see for example [Bal07] and [Bal09]). There is also a version of "simultaneous" Bertini's theorem for a pencil of hypersurfaces over finite fields [AG22].

Structure of the paper. The proof of the main theorem will be separated into two cases, according to whether $n \geq 3$ or $n = 2$. In Section 2, we handle the case $n \geq 3$ by employing a geometric argument that takes advantage of the fact that there exist nonblocking hypersurfaces in $\mathbb{P}^n$ containing a line (see Lemma 2.1). The proof for the $n = 2$ case is more novel, and requires a delicate choice of a pencil. In Section 3.1, we briefly discuss how the past results in the literature imply only certain special cases of our main theorem when $n = 2$. We then present a detailed proof of Theorem 1.1 when $n = 2$ in Section 3.2. Finally, in Section 4, we discuss how "efficient" a completely nonblocking pencil of curves can be.

2. Pencil of nonblocking hypersurfaces in high dimensions

The purpose of this section is to prove Theorem 1.1 for $n \geq 3$.

We start by proving the following auxiliary result which guarantees the existence of a hypersurface that is nonblocking and contains a fixed $\mathbb{F}_q$-line.

Lemma 2.1. *Let L be a fixed $\mathbb{F}_q$-line in $\mathbb{P}^n$ with $n \geq 3$. Given any $d \geq 2$, there exists a hypersurface $X \subset \mathbb{P}^n$ defined over $\mathbb{F}_q$ with degree d such that $L \subset X$ and X is not blocking.*

Proof. Let $x_0, x_1, ..., x_n$ be the homogeneous coordinates on $\mathbb{P}^n$. Without loss of generality, we can assume that $L = \{x_0 = x_1 = ... = x_{n-2} = 0\}$. Let X be a hypersurface defined by the equation $F(x_0, x_1, ..., x_n) = x_0^d + x_1 h(x_2, x_n)$, where $h(x_2, x_n)$ is a homogeneous polynomial of degree $d-1$ so that the specialized polynomial:

$$F|_{x_2=x_3=...=x_{n-1}=x_0, x_n=x_1} = x_0^d + x_1 h(x_0, x_1)$$

has no $\mathbb{F}_q$-roots $[x_0 : x_1]$ in $\mathbb{P}^1$. To see why such $h(x_2, x_n)$ exists, we can start with an *irreducible* binary form of degree d in x_0 and x_1 defined over $\mathbb{F}_q$,

$$x_0^d + a_1 x_0^{d-1} x_1 + a_2 x_0^{d-2} x_1^2 + ... + a_d x_1^d$$

and re-group the terms,

$$x_0^d + x_1(a_1x^{d-1} + a_2x^{d-2}y + ... + a_dy^{d-1})$$

and finally set $h(x_2, x_n) = a_1x_2^{d-1} + a_2x_2^{d-2}x_n + ... + a_{d-1}x_2x_n^{d-2} + a_dx_n^{d-1}$.

By construction, the hypersurface X contains the line L, since substituting $x_0 = x_1 = ... = x_{n-2} = 0$ makes the equation of F identically vanish. On the other hand, we claim that X is not blocking. Indeed, consider the intersection of X with the $\mathbb{F}_q$-line L_1 given by $\{x_2 = x_3 = ... = x_{n-1} = x_0\} \cap \{x_n = x_1\}$. The intersection $X \cap L_1$ is computed by specializing the defining equation $F(x_0, ..., x_n) = 0$ of the hypersurface by setting the variables $x_2, ..., x_{n-1}$ equal to x_0, and setting the variable x_n equal to x_1. By construction, this results in a binary form $x_0^d + x_1h(x_0, x_1)$ which has no $\mathbb{F}_q$-roots in $\mathbb{P}^1$. In particular, $X \cap L_1$ has no $\mathbb{F}_q$-points, and therefore X is not blocking. □

As another ingredient in our proof, we will rely on the following lemma regarding interpolation in algebraic geometry. We denote by V_d the vector space of homogeneous forms of degree d in $n+1$ variables $x_0, x_1, ..., x_n$. The projective space $\mathbb{P}(V_d)$ paramaterizes hypersurfaces of degree d in $\mathbb{P}^n$.

Lemma 2.2. *Fix a finite field $\mathbb{F}_q$, and consider any k distinct $\overline{\mathbb{F}_q}$-points $P_1, P_2, \ldots, P_k$ in $\mathbb{P}^n$. Let $W \subseteq V_d$ be the subspace (over $\overline{\mathbb{F}_q}$) corresponding to the hypersurfaces of degree d passing through $P_1, P_2, \ldots, P_k$. If $d \geq k-1$, then W has codimension k.*

Proof. The proof of this lemma has already appeared in the special case of plane curves $(n = 2)$ in our previous work [AGY22b, Proposition 3.1]. The same proof extends to the hypersurface case by replacing every occurrence of the word "line" in that proof with the word "hyperplane". We also mention that the result is known to the experts (see [Poo04, Lemma 2.1] for a proof using the cohomological language). □

We now proceed to the proof of the main theorem for $n \geq 3$.

Proof of Theorem 1.1 for $n \geq 3$. Fix an $\mathbb{F}_q$-line L, say $L = \{x_0 = x_1 = ... = x_{n-2} = 0\}$. We have a subspace of V_d given by:

$$V_{d,L} = \{f \in V_d \mid f \text{ identically vanishes on } L\}.$$

In other words, the polynomials in $V_{d,L}$ correspond to hypersurfaces which contain the $\mathbb{F}_q$-line L. The codimension of $V_{d,L}$ inside V_d is exactly $d+1$. This is because a homogeneous polynomial f of degree d vanishes along $L = \{x_0 = x_1 = ... = x_{n-2} = 0\}$ if and only if the $d+1$ coefficients in front of the monomials $x_{n-1}^i x_n^{d-i}$ (for $i = 0, 1, ..., d$) all vanish. These $d+1$ coefficients are coordinates in the parameter space, so we obtain that $\operatorname{codim}(V_{d,L}) = d+1$.

Now, let $Q_1 \in L(\mathbb{F}_{q^d})$ such that $Q_1 \notin L(\mathbb{F}_{q^r})$ for $r < d$. Consider the orbit of Q with its conjugates under the Frobenius map $[x_0 : ... : x_n] \mapsto [x_0^q : ... : x_n^q]$. This orbit forms a set $S = \{Q_1, Q_2, ..., Q_d\}$ of d distinct points that is invariant under the Frobenius action. By Lemma 2.2, we see that passing through these d points $Q_1, ..., Q_d$ impose linearly independent conditions in the space V_d. Thus, the vector subspace,

$$W = \{f \in V_d \mid f \text{ vanishes at all of } Q_1, Q_2, ..., Q_d\}$$

has codimension d inside V_d. Note that W is a subspace defined over $\mathbb{F}_q$, because $\{Q_1, Q_2, ..., Q_d\}$ forms a Galois orbit. It is also clear that $V_{d,L} \subseteq W$. Since $\dim(W) = \dim(V_{d,L}) + 1$, we see that $V_{d,L}$ is an $\mathbb{F}_q$-hyperplane inside W. After projectivizing, $\mathbb{P}(V_{d,L})$ is a hyperplane inside $\mathbb{P}(W)$.

By Lemma 2.1, there exists a nonblocking hypersurface $X \in V_{d,L}$ defined over $\mathbb{F}_q$. Consider the pencil $\mathcal{L} = \mathbb{P}^1$ spanned by the point X and other $Y \in W \setminus V_{d,L}$ such that Y is also defined over $\mathbb{F}_q$. By construction, $\mathcal{L}$ lies entirely inside $\mathbb{P}(W)$ and intersects $\mathbb{P}(V_{d,L})$ in exactly the point X. Now, we claim that each of the $q+1$ distinct members of $\mathcal{L}$ is nonblocking. This is true for the special hypersurface X by construction. The other q distinct $\mathbb{F}_q$-members of $\mathcal{L}$ are nonblocking hypersurfaces, because they intersect the $\mathbb{F}_q$-line $L = \{x_0 = x_1 = ... = x_{n-2} = 0\}$ in no $\mathbb{F}_q$-points. Indeed, they intersect L in the non-$\mathbb{F}_q$-points $Q_1, Q_2, ..., Q_d$ and there are no other points along L due to Bezout's theorem. We are able to apply Bezout's theorem because these q members of the pencil (other than X) do not contain the line L, since they are not in $\mathbb{P}(V_{d,L})$. □

3. Pencil of nonblocking plane curves

In this section, we will prove Theorem 1.1 in the case $n = 2$. Note that Lemma 2.1 fails trivially in this case, so we need to modify our approach.

3.1. **Comparison with past results.** As we will see in the next subsection, the proof of Theorem 1.1 is considerably more intricate for the plane curve case compared to the higher-dimensional case. We give more context to the difficulty of this problem. In particular, we explain how past results in the literature about blocking curves can only prove Theorem 1.1 in certain special cases.

Special Case 1. Suppose $\mathrm{char}(\mathbb{F}_q) \neq 3$ and $q > d^6$. There exists a smooth pencil $\mathcal{L}$ with degree d over $\mathbb{F}_q$ whenever $\mathrm{char}(\mathbb{F}_q) \neq 3$ by [AGR22, Theorem 2], and all $\mathbb{F}_q$-members in such a pencil are nonblocking curves provided that $q > d^6$ [AGY22a, Theorem 1.2].

Special Case 2. Suppose $\mathrm{char}(\mathbb{F}_q) = 3$ and $q > Cd^{12}$ for some absolute constant C. There exists a smooth pencil $\mathcal{L}$ with degree d over $\mathbb{F}_q$ [AG23, Theorem 1.3] provided that $q > Cd^{12}$, and all $\mathbb{F}_q$-members in such a pencil are nonblocking curves [AGY22a, Theorem 1.2].

Special Case 3. Suppose $d \geq 3(q-1)$. We can partition $\mathbb{P}^2(\mathbb{F}_q)$ into q sets with size q and a set with size $q+1$ such that not all $q+1$ points are collinear. Then none of these $q+1$ sets are blocking. By applying [AGY23, Proposition 2.1], we obtain a pencil $\mathcal{L}$ such that the sets of $\mathbb{F}_q$-points on the $q+1$ members induce the same partition. Thus, all $\mathbb{F}_q$-members in such a pencil $\mathcal{L}$ are nonblocking curves.

We mentioned in the introduction that statements (1) and (2) suggest that statement (3) is likely to be true. The special cases above can be regarded as consequences of statements (1) and (2). However, a proof of Theorem 1.1 in full generality cannot rely on statements (1) and (2).

3.2. **Proof of Theorem 1.1 for $n = 2$.** We will work with a fixed degree $d \geq 2$ over a finite field $\mathbb{F}_q$. The key idea is to find a pencil such that each $\mathbb{F}_q$-member is irreducible and has at most $q+2$ distinct $\mathbb{F}_q$-points. We break the proof into several steps.

Step 1 (Construction of the pencil). Consider the map $f_d \colon \mathbb{F}_q \to \mathbb{F}_q$ given by $f_d(x) = x^{2d-1} - x^{d-1}$. Since $f_d(0) = f_d(1) = 0$, the map f_d is not injective, and therefore not surjective. Let $\beta \in \mathbb{F}_q^*$ be any element not in the image of f_d. Let $g = 1/\beta \in \mathbb{F}_q^*$, and consider the pencil $\langle F_d, G_d \rangle$ where:

$$F_d(x,y,z) = y^d - z^d - z^{d-1}x, \qquad \text{and} \qquad G_d(x,y,z) = z^d + gy^{d-1}x.$$

We claim that the pencil $\mathcal{L} = \langle F_d, G_d \rangle$ satisfies the desired properties of Theorem 1.1, that is, all the $q+1$ curves defined over $\mathbb{F}_q$ in $\mathcal{L}$ are nonblocking.

Step 2 (Reduction to the low degree case). For each $[s : t] \in \mathbb{P}^1(\mathbb{F}_q)$, let $C_{d,[s:t]}$ denote the curve $sF_d - tG_d = 0$. In order to check that every $\mathbb{F}_q$-member $\mathcal{L}$ is a nonblocking curve, let us explain why it suffices to consider the case when $d \leq q$. Recall that a curve C is nonblocking if and only if $C(\mathbb{F}_q)$ is not a blocking set. Thus, it suffices to show $C_{d,[s:t]}(\mathbb{F}_q) = C_{q+d-1,[s:t]}(\mathbb{F}_q)$ for each $[s : t] \in \mathbb{P}^1(\mathbb{F}_q)$. This follows from the shape of F_d and G_d and the identity $b^{r+q-1} = b^r$ that holds for any $r \geq 1$ and $b \in \mathbb{F}_q$. If we can show that the conclusion holds for the pencil $\langle F_d, G_d \rangle$ of degree d, then it will also hold for the pencil $\langle F_{q+d-1}, G_{q+d-1} \rangle$ of degree $d + q - 1$. As a consequence, we are justified to assume $d \leq q$ for the rest of the proof.

Step 3 (Irreducibility of the curves). For every $[s : t] \in \mathbb{P}^1(\mathbb{F}_q)$, we claim that the polynomial

$$sF_d - tG_d = s(y^d - z^d + z^{d-1}x) - t(z^d + gy^{d-1}x) \tag{1}$$

is absolutely irreducible, that is, irreducible in $\overline{\mathbb{F}_q}[x, y, z]$. When $s = 0$ or $t = 0$, the result easily follows. We will assume $s \neq 0$ and $t \neq 0$. After scaling by $1/s$, it suffices to show that the polynomial

$$y^d - z^d + z^{d-1}x - t(z^d + gy^{d-1}x) \tag{2}$$

is irreducible in $\overline{\mathbb{F}_q}[x, y, z]$ for every $t \in \mathbb{F}_q^*$. Assume, to the contrary, that the polynomial in (2) splits nontrivially. Since it has degree 1 in x, we must have a factor $z - ay$ (using homogeneity of the polynomial in variables y and z) for some $a \in \overline{\mathbb{F}_q}$. Substituting $z = ay$ into the polynomial (2) and collecting coefficients of x and y^d, respectively, we obtain two relations:

$$a^{d-1} = tg \tag{3}$$

$$1 - a^d - ta^d = 0 \tag{4}$$

From (4), we get $a^d = \frac{1}{1+t}$ which implies $a^d \in \mathbb{F}_q^*$. Combining $a^d \in \mathbb{F}_q^*$ with (3), we deduce that $a \in \mathbb{F}_q^*$. On the other hand, (4) implies

$$t = \frac{1 - a^d}{a^d}. \tag{5}$$

Combining (3) and (5), we obtain

$$a^{d-1} = \frac{1 - a^d}{a^d} \cdot g \implies \frac{1}{g} = \frac{1 - a^d}{a^{2d-1}} = \left(\frac{1}{a}\right)^{2d-1} - \left(\frac{1}{a}\right)^{d-1}.$$

Therefore, $\frac{1}{g} = \beta$ is in the image of the function $f_d(x) = x^{2d-1} - x^{d-1}$. This contradicts the choice of β, and we conclude the (absolute) irreducibility of the polynomial in (1). We have established that each $\mathbb{F}_q$-member of the pencil $\langle F_d, G_d \rangle$ is an irreducible curve.

Step 4 (Counting the number of $\mathbb{F}_q$-points). Let $[s : t] \in \mathbb{P}^1(\mathbb{F}_q)$. We claim that there are at most $q + 2$ distinct $\mathbb{F}_q$-points on the curve defined by:

$$C_{d,[s:t]} : sF_d - tG_d = s \cdot (y^d - z^d + z^{d-1}x) - t \cdot (z^d + gy^{d-1}x) = 0.$$

We will now proceed to analyze several cases, depending on whether or not, s or t are zero.

When $t = 0$, we have the curve $C_0 := C_{d,[1:0]}$ defined by $y^d - z^d + z^{d-1}x = 0$. We claim that C_0 has $q + 1$ distinct $\mathbb{F}_q$-points. To count the number of $\mathbb{F}_q$-points of C_0, we consider two cases:

Case 1. $z \neq 0$. In this case, for any $(y, z) \in \mathbb{F}_q \times \mathbb{F}_q^*$, we can uniquely solve for $x \in \mathbb{F}_q$. For any $r \in \mathbb{F}_q^*$, the pair (ry, rz) results in rx, leading to the same point $[rx : ry : rz] = [x : y : z]$ in $\mathbb{P}^2$. Thus, there are $\frac{q\cdot(q-1)}{q-1} = q$ distinct $\mathbb{F}_q$-points on C_0 with $z \neq 0$.

Case 2. $z = 0$. In this case, $y = 0$ which means that $[x : y : z] = [1 : 0 : 0]$ is one additional $\mathbb{F}_q$-point in C_0.

When $s = 0$, we have the curve $C_\infty := C_{d,[0:1]}$ defined by $z^d + gy^{d-1}x = 0$. The similar analysis applies: when $y \neq 0$, we can solve uniquely for $x \in \mathbb{F}_q$, giving us a total of $\frac{q\cdot(q-1)}{q-1} = q$ points. When $y = 0$, we get an additional $\mathbb{F}_q$-point $[1 : 0 : 0]$ on C_∞. Thus, C_∞ has $q+1$ distinct $\mathbb{F}_q$-points as well.

Next, we focus on the case when $s \neq 0$ and $t \neq 0$. After scaling, it is enough to work with the curves defined by $C_{d,[1:t]}$ which we denote by C_t for simplicity. We have,

$$C_t\colon\ y^d - z^d + z^{d-1}x - t(z^d + gy^{d-1}x) = 0 \tag{6}$$

where $t \in \mathbb{F}_q^*$. We rewrite,

$$C_t\colon\ y^d - z^d - tz^d + (z^{d-1} - tgy^{d-1})x = 0. \tag{7}$$

To count the number of $\mathbb{F}_q$-points $[x : y : z]$ on C_t, we consider two cases:

Case 1. $z^{d-1} - tgy^{d-1} \neq 0$. In this case, the number of possible pairs is $(y, z) \in \mathbb{F}_q \times \mathbb{F}_q$ is at most $q^2 - 1$. For each such pair, we can solve for x uniquely in (7). Since (ry, rz) results in rx for any $r \in \mathbb{F}_q^*$, which corresponds to $[rx : ry : rz] = [x : y : z]$ in $\mathbb{P}^2$, the number of $\mathbb{F}_q$-points on C_t in this case is at most $\frac{q^2-1}{q-1} = q + 1$.

Case 2. $z^{d-1} - tgy^{d-1} = 0$. In this case, we must also have $y^d - z^d - tz^d = 0$. The only additional $\mathbb{F}_q$-point we get on C_t is $[1 : 0 : 0]$. Indeed, the analysis in **Step 3** shows that the only solution to the system of two equations:

$$y^d - z^d - tz^d = 0, \quad \text{and} \quad z^{d-1} - tgy^{d-1} = 0$$

in $\overline{\mathbb{F}_q} \times \overline{\mathbb{F}_q}$ is $(y, z) = (0, 0)$, for otherwise C_t is not geometrically irreducible.

We conclude that C_t has at most $q + 2$ distinct $\mathbb{F}_q$-points for every $t \in \mathbb{F}_q^*$.

Step 5 (Conclusion). Let $[s : t] \in \mathbb{P}^1(\mathbb{F}_q)$. We have shown that $C_{d,[s:t]}$ is geometrically irreducible and $|C_{d,[s:t]}(\mathbb{F}_q)| \leq q + 2$. For the sake of contradiction, suppose that $C_{d,[s:t]}$ is blocking; then $C_{d,[s:t]}(\mathbb{F}_q)$ must be a trivial blocking set, as otherwise $|C_{d,[s:t]}(\mathbb{F}_q)| \geq q + \sqrt{q} + 1$ [Bru71]. It follows that $C_{d,[s:t]}$ contains all the $q + 1$ distinct $\mathbb{F}_q$-points of a line L defined over $\mathbb{F}_q$. However, since $C_{d,[s:t]}$ is geometrically irreducible, $C_{d,[s:t]} \cap L$ has at most $d \leq q$ intersection points by Bézout's theorem, a contradiction. This completes the proof that all curves defined over $\mathbb{F}_q$ in the pencil $\mathcal{L} = \langle F_d, G_d \rangle$ are nonblocking.

Remark 3.1. If we allow the pencil to have at most one blocking curve, then the problem becomes much easier. Indeed, for each $d \geq 2$ and q, we can give an explicit construction of a "near miss" nonblocking pencil as follows. Let $h(t) \in \mathbb{F}_q[t]$ be an irreducible polynomial of degree d. Consider the pencil $\mathcal{L} = \langle F, G \rangle$ given by,

$$F(x, y, z) = x^d \qquad \text{and} \qquad G(x, y, z) = z^d h(y/z).$$

Then the $\mathbb{F}_q$-member corresponding to $F = 0$ is trivially blocking, but the other q distinct $\mathbb{F}_q$-members of $\mathcal{L}$ are not blocking (since their intersection with the line $x = 0$ has no $\mathbb{F}_q$-points).

4. EFFICIENT NONBLOCKING PENCILS

We have proved in Theorem 1.1 the existence of a pencil of plane curves whose $\mathbb{F}_q$-members are nonblocking, that is, every $\mathbb{F}_q$-member admits a *skew* $\mathbb{F}_q$-line (namely, a line which meets the curve at no $\mathbb{F}_q$-points). It is natural to ask how many skew $\mathbb{F}_q$-lines need to be present to ensure that the pencil is completely nonblocking. It is impossible to have two $\mathbb{F}_q$-lines L_1 and L_2 such that every $\mathbb{F}_q$-member is skew to either L_1 or L_2; indeed, the intersection point $P \in L_1 \cap L_2$ would be contained in some $\mathbb{F}_q$-member of the pencil. Next, we show that it is possible to have three $\mathbb{F}_q$-lines L_1, L_2, and L_3 such that every $\mathbb{F}_q$-member is skew to one of L_1, L_2, and L_3. In other words, for the "most efficient" pencil of nonblocking curves, three lines are sufficient to witness that all curves in the pencil are nonblocking.

We begin with the following criterion for a Fermat-type curve to be nonblocking.

Lemma 4.1. *Assume the characteristic of the field $\mathbb{F}_q$ is not 2. If $(q-1)/\gcd(d, q-1)$ is odd and $a, b, c \in \mathbb{F}_q^*$, then the curve C defined by*

$$ax^d + by^d + cz^d = 0$$

is nonblocking. Moreover, one of the three lines $x = 0$, $y = 0$, $z = 0$ is a skew line to C.

Proof. Let $d' = \gcd(q-1, d)$. Note d-th powers in $\mathbb{F}_q^*$ are essentially d'-th powers in $\mathbb{F}_q^*$. If the curve C meets $x = 0$, then $by^d + cz^d = 0$, which implies that $-b/c$ is a d'-th power. Similarly, if the curve C meets both $y = 0$ and $z = 0$, then $-c/a$ and $-a/b$ are also d'-th powers. In particular, $-1 = (-a/b) \cdot (-b/c) \cdot (-c/b)$ is a d'-th power, that is, $(-1)^{(q-1)/d'} = 1$, contradicting our assumption that $(q-1)/d'$ is odd and that $\mathrm{char}(\mathbb{F}_q) \neq 2$. □

Proposition 4.2. *Assume the characteristic of the field $\mathbb{F}_q$ is not 2. If $\gcd(q-1, d) \geq 3$ and $(q-1)/\gcd(q-1, d)$ is odd, then there exists a pencil of nonblocking curves of degree d over $\mathbb{F}_q$ witnessed by the lines $x = 0, y = 0$ and $z = 0$.*

Proof. Let $d' = \gcd(q-1, d)$. Since $(q-1)/d'$ is odd, it follows that -1 is not a d'-th power in $\mathbb{F}_q^*$. Since $d' \geq 3$, we can always pick an element r in $\mathbb{F}_q^*$ such that both r and $-r$ are *not* d'-th powers in $\mathbb{F}_q^*$.

Let $F = x^d + y^d$, and $G = y^d + rz^d$. Since -1 is not a d'-th power, the $\mathbb{F}_q$-line $z = 0$ is skew to the curve $\{F = 0\}$. Similarly, since $-r$ is not a d'-th power, the line $x = 0$ is skew to the curve $\{G = 0\}$. In particular, these two curves $\{F = 0\}$ and $\{G = 0\}$ are nonblocking. Moreoever, as r is not a d'-th power, the curve $\{F - G = 0\}$ defined by the polynomial $(x^d + y^d) - (y^d + rz^d) = x^d - rz^d$ is also nonblocking, since it admits $y = 0$ as a skew line. In fact, each of these 3 curves only has one $\mathbb{F}_q$-point.

Consider the pencil $\mathcal{L} = \langle F, G \rangle$. Other than the 3 special $\mathbb{F}_q$-members mentioned above, all other curves in $\mathcal{L}$ have the form $ax^d + by^d + cz^d = 0$, where a, b, c are nonzero. We can then apply Lemma 4.1 to get the desired conclusion. □

Remark 4.3. The above construction does not generalize to all q and $d \geq 2$. Indeed, it is known that when q is a square, the Hermitian curve $x^{\sqrt{q}+1} + y^{\sqrt{q}+1} + z^{\sqrt{q}+1} = 0$ is blocking [AGY22a, Example 1.5]. More generally, one can construct a family of Fermat-type curves which are Frobenius nonclassical and blocking [AGY22a, Section 4].

ACKNOWLEDGEMENTS

The second author is supported by an NSERC Discovery grant. The third author is supported by a doctoral fellowship from the University of British Columbia.

REFERENCES

[AG22] S. Asgarli and D. Ghioca, *A Bertini type theorem for pencils over finite fields*, Finite Fields Appl. **77** (2022), Paper No. 101936, 13.

[AG23] ______, *Smoothness in pencils of hypersurfaces over finite fields*, Bull. Aust. Math. Soc. **107** (2023), no. 1, 85–94.

[AGR22] S. Asgarli, D. Ghioca, and Z. Reichstein, *Linear families of smooth hypersurfaces over finitely generated fields*, to appear in Finite Fields Appl. (2022), available at https://arxiv.org/abs/2207.00918.

[AGY22a] S. Asgarli, D. Ghioca, and C. H. Yip, *Blocking sets arising from plane curves over finite fields*, arXiv e-prints (2022), available at https://arxiv.org/abs/2208.13299.

[AGY22b] ______, *Most plane curves over finite fields are not blocking*, arXiv e-prints (2022), available at https://arxiv.org/abs/2211.08523.

[AGY23] ______, *Proportion of blocking curves in a pencil*, arXiv e-prints (2023), available at https://arxiv.org/abs/2301.06019.

[Bal07] E. Ballico, *Bertini's theorem over a finite field for linear systems of quadrics*, Int. J. Pure Appl. Math. **35** (2007), no. 4, 453–455.

[Bal09] ______, *Vanishings and non-vanishings of homogeneous forms over a finite field*, Int. J. Pure Appl. Math. **57** (2009), no. 2, 219–224.

[Bru71] A. Bruen, *Blocking sets in finite projective planes*, SIAM J. Appl. Math. **21** (1971), 380–392.

[BSS14] A. Blokhuis, P. Sziklai, and T. Szőnyi, *Blocking sets in projective spaces*, Current research topics in galois geometry, 2014, pp. 63–86 (English).

[LW54] S. Lang and A. Weil, *Number of points of varieties in finite fields*, Amer. J. Math. **76** (1954), 819–827.

[Poo04] B. Poonen, *Bertini theorems over finite fields*, Ann. of Math. (2) **160** (2004), no. 3, 1099–1127.

[SS98] P. Sziklai and T. Szőnyi, *Blocking sets and algebraic curves*, Rend. Circ. Mat. Palermo (2) Suppl. **51** (1998), 71–86.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, SANTA CLARA UNIVERSITY, 500 EL CAMINO REAL, USA 95053

Email address: sasgarli@scu.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, 1984 MATHEMATICS ROAD, CANADA V6T 1Z2

Email address: dghioca@math.ubc.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, 1984 MATHEMATICS ROAD, CANADA V6T 1Z2

Email address: kyleyip@math.ubc.ca