A VARIANT OF THE SIEGEL'S THEOREM FOR DRINFELD MODULES

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ABSTRACT. We complete the proof of a Siegel type statement for finitely generated Φ-submodules of $\mathbb{G}_a$ under the action of a Drinfeld module Φ.

1. INTRODUCTION

In 1929, Siegel ([Sie29]) proved that if C is an irreducible affine curve defined over a number field K and C has at least three points at infinity, then there are at most finitely many K-rational points on C that have integral coordinates. The most important two ingredients in the proof of Siegel's theorem are diophantine approximation, along with the fact that certain groups of rational points are finitely generated; when C has genus greater than 0, the group in question is the Mordell-Weil group of the Jacobian of C, while when C has genus 0, the group in question is the group of S-units in a finite extension of K. Motivated by the analogy between rank 2 Drinfeld modules and elliptic curves (along with the understanding that higher rank Drinfeld modules of generic characteristic are the right vehicle in characteristic p for similar conjectures to Diophantine questions one would pose for abelian varieties in characteristic 0), Ghioca and Tucker conjectured in [GT08a] a Siegel type statement for finitely generated Φ-submodules Γ of $\mathbb{G}_a(K)$ (where Φ is a generic characteristic Drinfeld module of arbitrary rank and K is a finite extension of $\mathbb{F}_p(t)$). In [GT07], Ghioca and Tucker proved the Siegel type statement for Drinfeld modules under the technical hypothesis that the ground field K admits a single place which lies over the place at infinity of $\mathbb{F}_p(t)$. In the current paper, we are able to remove this technical hypothesis on the field K and prove the following general result for Drinfeld modules in the spirit of the famous Siegel's theorem.

Theorem 1.1. *Let q be a power of the prime number p, let K be a finite extension of the function field $\mathbb{F}_q(t)$ and let Φ be a Drinfeld module of generic characteristic defined over K. Let Γ be a finitely generated Φ-submodule of $\mathbb{G}_a(K)$, let $\alpha \in K$, and let S be a finite set of places of the field K. Then there are finitely many $\gamma \in \Gamma$ such that γ is S-integral with respect to α.*

We refer the reader to Section 2 for more details on Drinfeld modules, including the notion of S-integral points with respect to a given point.

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The strategy of our proof is identical with the strategy employed in [GT07]; actually, with one exception, the statements proven in [GT07] are valid in the generality of our Theorem 1.1. However, the main result from the aforementioned paper, i.e., [GT07, Proposition 3.12] is proven under the technical assumption that there exists a single place in the function field K lying above the place at infinity from $\mathbb{F}_q(t)$. Moreover, the strategy of proof from [GT07, Proposition 3.12] does not extend to arbitrary function fields K, as explained in [GT07, Remark 3.14]. So, the main result of our current paper is to develop an alternative strategy for proving [GT07, Proposition 3.12] for arbitrary function fields K; the statement generalizing [GT07, Proposition 3.12] is proven in our Proposition 3.6. Finally, we note that in Theorem 1.1, if Φ were a Drinfeld module of special characteristic, then its conclusion could fail (as can be easily seen in the case the Drinfeld module is simply given by $\Phi_t(x) := x^q$).

Our Theorem 1.1 completes the proof of the Siegel's theorem in the context of Drinfeld modules. Over the past 30 years, there was a significant increase in the study of the arithmetic of Drinfeld modules which established the validity of several classical theorems from the arithmetic geometry of abelian varieties in the context of Drinfeld modules. Indeed, Scanlon [Sca02] proved the Manin-Mumford type theorem for Drinfeld modules, conjectured by Denis [Den92a]. The second author proved an equidistribution statement for torsion points of Drinfeld modules, which may be interpreted as a weaker variant of the classical equidistribution theorem of Szpiro-Ullmo-Zhang [SUZ97] for torsion points of abelian varieties. Breuer [Bre05] proved an André-Oort type theorem for Drinfeld modules in the spirit of the classical results of Edixhoven-Yafaev [EY03] for Shimura varieties. The second author (both in a single author paper [Ghi05] and also in a joint paper with Tucker [GT08b]) proved various instances of a Mordell-Lang type statement for Drinfeld modules conjectured by Denis [Den92a]. Actually, the paper [GT08b] constituted the starting point of the Dynamical Mordell-Lang Conjecture (formulated in [GT09]), which by itself generated extensive research in the past 15 years (for a comprehensive discussion of the Dynamical Mordell-Lang Conjecture, see [BGT06], especially [BGT06, Chapter 12] which details the connection between the aforementioned conjecture and Denis' conjecture for Drinfeld modules from [Den92a]).

The plan of our paper is as follows: in Section 2 we give the basic definitions and notation, and then, in Section 3 we prove the main results, which is Proposition 3.6 and then derive Theorem 1.1 as a consequence of our Proposition 3.6 and of the previous results established in [GT07].

2. Notation

Our notational section has a significant overlap with [GT07, Section 2].

2.1. **General notation.** We let $\mathbb{N}_0$ denote the non-negative integers: $\{0, 1, \dots\}$, i.e., $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2.2. **Drinfeld modules.** We begin by defining a Drinfeld module. Let p be a prime and let q be a power of p. Let $A := \mathbb{F}_q[t]$, let K be a finite field extension of $\mathbb{F}_q(t)$, and let $\overline{K}$ be an algebraic closure of K. We let τ be the Frobenius on $\mathbb{F}_q$, and we extend its action on $\overline{K}$. Let $K\{\tau\}$ be the ring of polynomials in τ with coefficients from K (the addition is the usual addition, while the multiplication is the composition of functions).

A Drinfeld module is a morphism $\Phi : A \to K\{\tau\}$ for which the coefficient of τ^0 in $\Phi(a) =: \Phi_a$ is a for every $a \in A$, and there exists $a \in A$ such that $\Phi_a \neq a\tau^0$. The definition given here represents what Goss [Gos96] calls a Drinfeld module of "generic characteristic".

We note that usually, in the definition of a Drinfeld module, A is the ring of functions defined on a projective nonsingular curve C, regular away from a closed point $\eta \in C$. For our definition of a Drinfeld module, $C = \mathbb{P}^1_{\mathbb{F}_q}$ and η is the usual point at infinity on $\mathbb{P}^1$. On the other hand, every ring of regular functions A as above contains $\mathbb{F}_q[t]$ as a subring, where t is a nonconstant function in A. Furthermore, even for such a general ring of regular functions A, we have that A is a finite $\mathbb{F}_q[t]$-module, which means that the statement of our Theorem 1.1 is left unchanged.

For every field extension $K \subset L$, the Drinfeld module Φ induces an action on $\mathbb{G}_a(L)$ by $a * x := \Phi_a(x)$, for each $a \in A$. We call Φ-*submodules* subgroups of $\mathbb{G}_a(\overline{K})$ which are invariant under the action of Φ. We define the *rank* of a Φ-submodule Γ be

$$\dim_{\mathrm{Frac}(A)} \Gamma \otimes_A \mathrm{Frac}(A).$$

A point α is *torsion* for the Drinfeld module action if and only if there exists $Q \in A \setminus \{0\}$ such that $\Phi_Q(\alpha) = 0$. The monic polynomial Q of minimal degree which satisfies $\Phi_Q(\alpha) = 0$ is called the *order* of α. Since each polynomial Φ_Q is separable, the torsion submodule Φ_{tor} lies in the separable closure K^{sep} of K.

2.3. **The coefficients of the Drinfeld module.** Each Drinfeld module is isomorphic over $\overline{K}$ to a Drinfeld module for which the leading coefficient of Φ_t equals 1; in particular, for each $a \in \mathbb{F}_q[t] \setminus \mathbb{F}_q$, we would then have that Φ_a is a polynomial whose leading coefficient lives in $\mathbb{F}_q$. This is a standard observation used previously in [Ghi07a, Ghi07b].

2.4. **Valuations and Weil heights.** Let $M_{\mathbb{F}_q(t)}$ be the set of places on $\mathbb{F}_q(t)$. We denote by v_∞ the place in $M_{\mathbb{F}_q(t)}$ such that $v_\infty(\frac{f}{g}) = \deg(g) - \deg(f)$ for every nonzero $f, g \in A = \mathbb{F}_q[t]$. We let M_K be the set of valuations on K. Then M_K is a set of valuations which satisfies a product formula (see [Ser97, Chapter 2]). Thus (with an appropriate normalization for each absolute value $|\cdot|_v$) we have

- for each nonzero $x \in K$, there are finitely many $v \in M_K$ such that $|x|_v \neq 1$; and
- for each nonzero $x \in K$, we have $\prod_{v \in M_K} |x|_v = 1$.

We may use these valuations to define a Weil height for each $x \in K$ as

$$h(x) = \sum_{v \in M_K} \log \max(|x|_v, 1). \tag{2.0.1}$$

Definition 2.1. *Each place in M_K which lies over v_∞ is called an infinite place. Each place in M_K which does not lie over v_∞ is called a finite place.*

2.5. **Canonical heights.** Let $\Phi : A \to K\{\tau\}$ be a Drinfeld module of *rank* d (i.e. the degree of Φ_t as a polynomial in τ equals d). The canonical height of $\beta \in K$ relative to Φ (see [Den92b]) is defined as

$$\widehat{h}(\beta) = \lim_{n \to \infty} \frac{h(\Phi_{t^n}(\beta))}{q^{nd}}.$$

Denis [Den92b] showed that a point is torsion if and only if its canonical height equals 0.

For every $v \in M_K$, we let the local canonical height of $\beta \in K$ at v be

$$\widehat{h}_v(\beta) = \lim_{n \to \infty} \frac{\log \max(|\Phi_{t^n}(\beta)|_v, 1)}{q^{nd}}. \tag{2.1.1}$$

Furthermore, for every $a \in \mathbb{F}_q[t]$, we have $\widehat{h}_v(\Phi_a(x)) = \deg(\Phi_a) \cdot \widehat{h}_v(x)$. It is clear that $\widehat{h}_v$ satisfies the triangle inequality, and also that $\sum_{v \in M_K} \widehat{h}_v(\beta) = \widehat{h}(\beta)$ (therefore, also $\widehat{h}(\cdot)$ satisfies the triangle inequality).

2.6. **Integrality and reduction.** Since we can always replace K by a finite extension, we will define the notion of S-integrality with respect to a given point α under the assumption that $\alpha \in K$ (note that we are interested in Theorem 1.1 for the S-integrality with respect to α within a given finitely generated Φ-module Γ and therefore, once again at the expense of replacing K by a finite extension, we may assume $\Gamma \subset K$).

Definition 2.2. *For a finite set of places $S \subset M_K$ and $\alpha \in K$, we say that $\beta \in K$ is S-integral with respect to α if for every place $v \notin S$, the following are true:*

- *if $|\alpha|_v \le 1$, then $|\alpha - \beta|_v \ge 1$.*
- *if $|\alpha|_v > 1$, then $|\beta|_v \le 1$.*

We note that if β is S-integral with respect to α, then it is also S'-integral with respect to α, where S' is a finite set of places containing S. Furthermore (as noted in [GT07, Subsection 2.6]), the notion of S-integrality with respect to a point is invariant when replacing K by a finite extension L and also replacing S by the set of places in L lying above the places from S.

Definition 2.3. *The Drinfeld module Φ has good reduction at a place v if for each nonzero $a \in A$, all coefficients of Φ_a are v-adic integers and the leading coefficient of Φ_a is a v-adic unit. If Φ does not have good reduction at v, then we say that Φ has bad reduction at v.*

It is immediate to see that Φ has good reduction at v if and only if all coefficients of Φ_t are v-adic integers, since we already assumed that the leading coefficient of Φ_t equals 1.

3. Proofs of our main results

Before proceeding to the proof of Theorem 1.1, we prove several facts about local heights. We start by stating a result obtained in [GT07, Lemma 3.11].

Proposition 3.1. *Suppose that Γ is a torsion-free Φ-submodule of $\mathbb{G}_a(K)$ generated by elements $\gamma_1, \dots, \gamma_r$. For each $i \in \{1, \dots, r\}$ let $(P_{n,i})_{n\in\mathbb{N}} \subset \mathbb{F}_q[t]$ be a sequence of polynomials such that for each $m \neq n$, the r-tuples $(P_{n,i})_{1\le i\le r}$ and $(P_{m,i})_{1\le i\le r}$ are distinct. Then for every $v \in M_K$, we have*

$$\liminf_{n\to\infty} \frac{\log |\sum_{i=1}^r \Phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d\deg P_{n,i}}} \ge 0. \tag{3.1.1}$$

From now on, let $\Phi_t = \sum_{i=0}^d a_i\tau^i$; also, as explained in Section 2, we assume from now on that $a_d = 1$. The following result is a special case of [GT07, Fact 3.6] (in the case Φ_t is monic).

Lemma 3.2. *For every place v of K, there exists a real number $M_v \ge 1$ such that for each $x \in K$, if $|x|_v > M_v$, then for every nonzero $Q \in A$, we have $|\Phi_Q(x)|_v = |x|_v^{q^{d\cdot\deg(Q)}}$. Moreover, if $|x|_v > M_v$, then $\widehat{h}_v(x) = \log|x|_v$.*

The following result is an immediate consequence of Lemma 3.2.

Lemma 3.3. *For each place v of M_K and for each $x \in K$, if $\widehat{h}_v(x) > 0$, then for all polynomials $Q \in \mathbb{F}_q[t]$ of sufficiently large degree, we have that $\widehat{h}_v(x) = \frac{\log|\Phi_Q(x)|_v}{q^{d\cdot\deg(Q)}}$.*

Proof. The proof is an immediate corollary of Lemma 3.2 once we note that if $\widehat{h}_v(x) > 0$ then there exists a nonzero integer ℓ (depending on x and v) such that $|\Phi_{t^\ell}(x)|_v > M_v$; moreover, we may assume ℓ is minimal with this property. Then Lemma 3.2 yields that

$$\widehat{h}_v(x) = \frac{\widehat{h}_v\left(\Phi_{t^\ell}(x)\right)}{q^{d\ell}} = \frac{\log|\Phi_{t^\ell}(x)|}{q^{d\ell}}. \tag{3.3.1}$$

Moreover, for each polynomial $Q \in \mathbb{F}_q[t]$ of degree at least equal to ℓ, we have that

$$|\Phi_Q(x)|_v = |\Phi_{t^\ell}(x)|_v^{q^{d(\deg(Q)-\ell)}}. \tag{3.3.2}$$

Equations (3.3.1) and (3.3.2) finish the proof of Lemma 3.3. □

The following result is an immediate consequence of [Ghi07a, Theorem 4.5] (which provides a more general positive lower bound for the canonical height of non-torsion points $x \in \overline{K}$ depending only on the number of places of bad reduction for the given Drinfeld module in the field extension $K(x)$).

Lemma 3.4. *There exists a positive constant c_0 such that for all non-torsion points $x \in K$, we have $\widehat{h}(x) \geq c_0$.*

The next result will be used in the proof of Proposition 3.6 (which is our main technical ingredient for the proof of Theorem 1.1).

Lemma 3.5. *Let $r \in \mathbb{N}$, let $\gamma_1, \dots, \gamma_r \in K$ and let $v \in M_K$. Assume that*

$$\widehat{h}_v(\gamma_1) > \frac{\max_{i>1} \widehat{h}_v(\gamma_i)}{q^d}. \tag{3.5.1}$$

Then there exists $n_0 \in \mathbb{N}$ (depending only on Φ, v and on $\gamma_1, \dots, \gamma_r$) and there exists a positive real number c_1 (depending only on Φ and v) such that for all polynomials $P_1, \dots, P_r \in \mathbb{F}_q[t]$ satisfying

(i) $\deg(P_1) > \max\{\deg(P_2), \cdots, \deg(P_r)\}$, *and*
(ii) $\min\{\deg(P_2), \cdots, \deg(P_r)\} \geq n_0$,

we have $\log |\Phi_{P_1}(\gamma_1) + \cdots + \Phi_{P_r}(\gamma_r)|_v > c_1 \cdot q^{d \cdot (\deg(P_1) - n_0)}$.

Proof. Let $M_v \geq 1$ be the real number as in the conclusion of Lemma 3.2 and let $L_v := 2M_v$. Then for each $i = 2, \dots, r$ such that $\widehat{h}_v(\gamma_i) = 0$, we must have that for all polynomials $Q_i \in \mathbb{F}_q[t]$ then

$$|\Phi_{Q_i}(\gamma_i)|_v < L_v. \tag{3.5.2}$$

Similarly, according to Lemma 3.3, we have that for each $i = 1, \dots, r$ for which $\widehat{h}_v(\gamma_i) > 0$ (note that our hypothesis yields that $\widehat{h}_v(\gamma_1) > 0$) then for all polynomials $Q_i \in \mathbb{F}_q[t]$ of degree larger than some positive integer n_0 (which depends only on v and on the γ_i's),

$$\log |\Phi_{Q_i}(\gamma_i)|_v = q^{d \deg(Q_i)} \cdot \widehat{h}_v(\gamma_i). \tag{3.5.3}$$

Furthermore, for each i as in equation (3.5.3), as long as $\deg(Q_i)$ is sufficiently large, then we have that

$$|\Phi_{Q_i}(\gamma_i)|_v \geq L_v > M_v. \tag{3.5.4}$$

Combining equations (3.5.2), (3.5.3) and (3.5.4), coupled with our hypotheses (3.5.1) and (i) from Lemma 3.5, we obtain that $|\Phi_{P_1}(\gamma_1)|_v > \max_{i>1} |\Phi_{P_i}(\gamma_i)|_v$ and therefore,

$$\left| \sum_{i=1} \Phi_{P_i}(\gamma_i) \right|_v = |\Phi_{P_1}(\gamma_1)|_v > M_v. \tag{3.5.5}$$

Finally, combining (3.5.5) with Lemma 3.2 (see also equation (3.5.3)), we obtain that

$$\log \left| \sum_{i=1}^{r} \Phi_{P_i}(\gamma_i) \right|_v = \log |\Phi_{P_1}(\gamma_1)|_v = q^{d \cdot (\deg(P_1) - n_0)} \cdot \log |\Phi_{t^{n_0}}(\gamma_1)|_v .$$

Since $|\Phi_{t^{n_0}}(\gamma_1)|_v \geq L_v > M_v$ and therefore, letting $c_1 := \log(2L_v/3) \geq \log(4/3) > 0$, we obtain the desired conclusion in Lemma 3.5. ∎

The following proposition is the key technical result required to prove Theorem 1.1 and it is the generalization of [GT07, Proposition 3.12] in the case of arbitrary function fields K.

Proposition 3.6. *Let Γ be a torsion-free Φ-submodule of $\mathbb{G}_a(K)$ generated by elements $\gamma_1, \dots, \gamma_r$. For each $i \in \{1, \dots, r\}$ let $(P_{n,i})_{n\in\mathbb{N}} \subset \mathbb{F}_q[t]$ be a sequence of polynomials such that for each $m \neq n$, the r-tuples $(P_{n,i})_{1\leq i\leq r}$ and $(P_{m,i})_{1\leq i\leq r}$ are distinct. Then there exists a place $v \in M_K$ such that*

$$\limsup_{n\to\infty} \frac{\log |\sum_{i=1}^r \Phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d\deg P_{n,i}}} > 0. \tag{3.6.1}$$

Proof. The hypothesis on the r-uples $(P_{n,i})_{1\leq i\leq r}$ implies that

$$\lim_{n\to\infty} \sum_{i=1}^r q^{d\deg P_{n,i}} = +\infty.$$

Combining this with the triangle inequality for the v-adic norm, we have that the sought statement is equivalent to the existence of a place v such that

$$\limsup_{n\to\infty} \frac{\log |\sum_{i=1}^r \Phi_{P_{n,i}}(\gamma_i)|_v}{\sum_{i=1}^r q^{d\deg P_{n,i}}} > 0. \tag{3.6.2}$$

Observe that it is sufficient to prove it for a subsequence $(n_k)_{k\geq 1} \subset \mathbb{N}$, since passing to a subsequence can only lower the lim sup. We will repeatedly use this fact during the proof and we will drop the extra indexes in order to lighten the notation.

The proof proceeds by induction on r. If $r = 1$, since γ_1 is non-torsion, there exists a place v such that $\widehat{h}_v(\gamma_1) > 0$ and the conclusion follows from Lemma 3.3. We can then assume that (3.6.2) holds true for all Φ-submodules of rank less than r and we are going to prove it for all Φ-submodules of rank r.

Let S_0 be the set of places $v \in M_K$ such that $\widehat{h}_v(\gamma) > 0$ for some $\gamma \in \Gamma$. This set is finite, as proved in [GT07, Fact 3.13].

If there exists j such that

$$\lim_{n\to\infty} \frac{q^{d\deg P_{n,j}}}{\sum_{i=1}^r q^{d\deg P_{n,i}}} = 0 \tag{3.6.3}$$

then the conclusion follows from the inductive hypothesis. Indeed, by the inductive hypothesis, there exists $v \in S_0$ such that

$$\limsup_{n\to\infty} \frac{\log |\sum_{i\neq j} \Phi_{P_{n,i}}(\gamma_i)|_v}{\sum_{i\neq j} q^{d\deg P_{n,i}}} > 0,$$

and combining this with (3.6.3) gives

$$\limsup_{n\to\infty} \frac{\log |\sum_{i\neq j} \Phi_{P_{n,i}}(\gamma_i)|_v}{\sum_{i=1}^r q^{d\deg P_{n,i}}} > 0. \tag{3.6.4}$$

We distinguish two cases. If $\widehat{h}_v(\gamma_j) = 0$, then $\{|\Phi_Q(\gamma_j)|_v\}_{Q\in\mathbb{F}_q[t]}$ is bounded, so that for large enough n we have

$$\left|\sum_{i\neq j}\Phi_{P_{n,i}}(\gamma_i)\right|_v = \left|\sum_{i=1}^{r}\Phi_{P_{n,i}}(\gamma_i)\right|_v$$

and the result follows from (3.6.4).

If $\widehat{h}_v(\gamma_j) > 0$, we apply Lemma 3.3 to get

$$\lim_{n\to\infty}\frac{\log|\Phi_{P_{n,j}}(\gamma_j)|_v}{\sum_{i=1}^{r} q^{d\deg P_{n,i}}} = \widehat{h}_v(\gamma_j)\cdot\lim_{n\to\infty}\frac{q^{d\deg P_{n,j}}}{\sum_{i=1}^{r} q^{d\deg P_{n,i}}} = 0.$$

This, combined with (3.6.4), gives

$$\left|\sum_{i\neq j}\Phi_{P_{n,i}}(\gamma_i)\right|_v = \left|\sum_{i=1}^{r}\Phi_{P_{n,i}}(\gamma_i)\right|_v \tag{3.6.5}$$

for large enough n. We then derive (3.6.2) using (3.6.4).

We therefore assume that there is no j for which (3.6.4) holds. Equivalently, there exists $B \geq 1$ such that for any n we have

$$\frac{\max_{1\leq i\leq r} q^{d\deg P_{n,i}}}{\min_{1\leq i\leq r} q^{d\deg P_{n,i}}} \leq B,$$

which is the same as

$$\max_{1\leq i\leq r}\deg P_{n,i} - \min_{1\leq i\leq r}\deg P_{n,i} \leq \frac{\log_q B}{d}. \tag{3.6.6}$$

We will proceed doing analysis at the places $v \in S_0$, which is non-empty since all γ_i are non-torsion.

The strategy of the proof goes as follows: if we cannot prove directly (3.6.2), then we find $\delta_1,\dots,\delta_r \in \Gamma$, a subsequence $(n_k)_{k\geq 1}$ and a sequence of r-uples of polynomials $(R_{k,i})_{1\leq i\leq r}$ such that

$$\sum_{i=1}^{r}\Phi_{P_{n_k,i}}(\gamma_i) = \sum_{i=1}^{r}\Phi_{R_{k,i}}(\delta_i) \quad\text{and}$$

$$0 < \liminf_{k\to\infty}\frac{\sum_{i=1}^{r} q^{d\deg P_{n_k,i}}}{\sum_{i=1}^{r} q^{d\deg R_{k,i}}} \leq \limsup_{k\to\infty}\frac{\sum_{i=1}^{r} q^{d\deg P_{n_k,i}}}{\sum_{i=1}^{r} q^{d\deg R_{k,i}}} < +\infty.$$

Thanks to these two relations, we have that if there exists a place v for which

$$\limsup_{k\to\infty}\frac{\log|\sum_{i=1}^{r}\Phi_{R_{k,i}}(\delta_i)|_v}{\sum_{i=1}^{r} q^{d\deg R_{k,i}}} > 0,$$

then also

$$\limsup_{k\to\infty}\frac{\log|\sum_{i=1}^{r}\Phi_{P_{n_k,i}}(\gamma_i)|_v}{\sum_{i=1}^{r} q^{d\deg P_{n_k,i}}} > 0.$$

In this way we reduce to proving (3.6.2) for the δ_i's and the sequence of r-uples of polynomials $(R_{k,i})_{1\leq i\leq r}$. We will choose them in such a way that the process cannot go on forever, thereby showing that (3.6.2) has to hold at a finite step for a suitable place v.

We claim that, if there exist j and v such that

$$\widehat{h}_v(\gamma_j) > \frac{1}{q^d}\max_{i\neq j}\widehat{h}_v(\gamma_i)$$

and

$$\deg P_{n,j} > \max_{i\neq j}\deg P_{n,i}$$

for all n big enough, then inequality (3.6.2) holds for the place v. Indeed, using (3.6.6), we have that for any i the degrees $\deg P_{n,i}$ go to infinity as $n\to\infty$. This, together with the two inequalities above, allows us to apply Lemma 3.5 to obtain (for a suitable positive integer n_0) that

$$\limsup_{n\to\infty}\frac{\log|\sum_{i=1}^r\Phi_{P_{n,i}}(\gamma_i)|_v}{\sum_{i=1}^r q^{d\deg P_{n,i}}} > c_1q^{-dn_0}\cdot\limsup_{n\to\infty}\frac{q^{d\deg P_{n,j}}}{\sum_{i=1}^r q^{d\deg P_{n,i}}} > 0,$$

where we used again (3.6.6).

In particular, if there exists j such that

$$\deg P_{n,j} > \max_{i\neq j}\deg P_{n,i}$$

for all n big enough, either (3.6.2) follows, or we can assume that for *all* places v we have

$$\widehat{h}_v(\gamma_j) \le \frac{1}{q^d}\max_{i\neq j}\widehat{h}_v(\gamma_i) \le \frac{1}{q^d}\max_{1\le i\le r}\widehat{h}_v(\gamma_i).$$

We will now describe the construction of the δ_i's and $R_{n,i}$'s we referred to before. In the first step of our process, for all n and each $i>1$, we divide (with quotient and remainder) $P_{n,i}$ by $P_{n,1}$, obtaining

$$P_{n,i} = P_{n,1}\cdot C_{n,i} + R_{n,i},$$

so that $\deg R_{n,i} < \deg P_{n,1}$; also, we note that $\deg R_{n,i}\le\deg P_{n,i}$. We also let $R_{n,1} := P_{n,1}$. From (3.6.6) the degrees $\deg C_{n,i}$ are uniformly bounded as $n\to\infty$. It follows that there are only finitely many possible polynomials $C_{n,i}$, so that, passing to a subsequence $(n_k)_{k\ge 1}$, we may assume that there exist polynomials C_i satisfying

$$C_{n,i} = C_i$$

for all n (where we dropped the index k of the subsequence).

Let

$$\delta_i = \begin{cases}\gamma_1 + \sum_{j=2}^r\Phi_{C_j}(\gamma_j) & \text{if } i=1\\ \gamma_i & \text{otherwise.}\end{cases}$$

Observe that for each n, we have

$$\sum_{i=1}^r\Phi_{P_{n,i}}(\gamma_i) = \sum_{i=1}^r\Phi_{R_{n,i}}(\delta_i), \tag{3.6.7}$$

as it follows from the definition of the δ_i's and of the $R_{n,i}$'s.

Also, since $\deg R_{n,i} \le \deg P_{n,i}$ for all n and i, we have

$$\sum_{i=1}^{r} q^{d\deg R_{n,i}} \le \sum_{i=1}^{r} q^{d\deg P_{n,i}},$$

which implies

$$0 < \liminf_{n\to\infty} \frac{\sum_{i=1}^{r} q^{d\deg P_{n,i}}}{\sum_{i=1}^{r} q^{d\deg R_{n,i}}}.$$

Using that $R_{n,1} = P_{n,1}$ we obtain

$$\begin{aligned}\frac{\sum_{i=1}^{r} q^{d\deg P_{n,i}}}{\sum_{i=1}^{r} q^{d\deg R_{n,i}}} &= \frac{\sum_{i=1}^{r} q^{d\deg P_{n,i}}}{q^{d\deg P_{n,1}}} \cdot \frac{q^{d\deg P_{n,1}}}{q^{d\deg P_{n,1}} + \sum_{i=2}^{r} q^{d\deg R_{n,i}}} \\ &\le \frac{\sum_{i=1}^{r} q^{d\deg P_{n,i}}}{q^{d\deg P_{n,1}}} \\ &\le \frac{rq^{d\max_i \deg P_{n,i}}}{q^{d\deg P_{n,1}}} \\ &\le rB,\end{aligned}$$

where we used (3.6.6) in the last step. In particular we derive

$$0 < \liminf_{n\to\infty} \frac{\sum_{i=1}^{r} q^{d\deg P_{n,i}}}{\sum_{i=1}^{r} q^{d\deg R_{n,i}}} \le \limsup_{n\to\infty} \frac{\sum_{i=1}^{r} q^{d\deg P_{n,i}}}{\sum_{i=1}^{r} q^{d\deg R_{n,i}}} < +\infty. \tag{3.6.8}$$

As explained before, thanks to (3.6.7) and (3.6.8), we can reduce the problem to the study of the δ_i's and the sequence of r-uples of polynomials $(R_{n,i})_{1\le i\le r}$.

In order for our strategy to work, we will also need that (3.6.6) has to hold for the polynomials $R_{n,i}$, that is, we want

$$\max_{1\le i\le r} \deg R_{n,i} - \min_{1\le i\le r} \deg R_{n,i} \le \frac{\log_q B'}{d}.$$

for a suitable constant B'. However, arguing as we did from (3.6.3) to (3.6.5), if the above is not satisfied, then there exists a place w such that

$$\limsup_{n\to\infty} \frac{\log |\sum_{i=1}^{r} \Phi_{R_{n,i}}(\delta_i)|_w}{\sum_{i=1}^{r} q^{d\deg R_{n,i}}} > 0.$$

Using (3.6.7) and (3.6.8) we obtain

$$\limsup_{n\to\infty} \frac{\log |\sum_{i=1}^{r} \Phi_{P_{n,i}}(\gamma_i)|_w}{\sum_{i=1}^{r} q^{d\deg P_{n,i}}} > 0,$$

thereby proving the statement. We may then assume that (3.6.6) holds also for the $R_{n,i}$'s (possibly with a different constant B).

Since

$$\deg R_{n,1} > \max_{i>1} \deg R_{n,i}$$

for all n, as we previously observed, if there exists a place v such that

$$\widehat{h}_v(\delta_1) > \frac{1}{q^d}\max_{i>1}\widehat{h}_v(\delta_i),$$

then (3.6.2) holds at the place v. Therefore, we may assume that for all places v we have

$$\widehat{h}_v(\delta_1) \le \frac{1}{q^d}\max_{i>1}\widehat{h}_v(\delta_i),$$

from which we get

$$\widehat{h}_v(\delta_1) \le \frac{1}{q^d}\max_{1\le i\le r}\widehat{h}_v(\gamma_i), \tag{3.6.9}$$

where we used that $\delta_i = \gamma_i$ for $i > 1$.

We now repeat the above construction of the δ_i's and $R_{n,i}$'s using $R_{n,2}$ in place of $P_{n,1}$, that is, we will proceed dividing by $R_{n,2}$ each of the $R_{n,i}$'s for $i \ne 2$. We will still denote the new polynomials with the letter R to not cluster the notation. We will then find a subsequence of $\mathbb{N}$, polynomials $R_{n,i}$ and δ_2 (given by a suitable Φ-linear combination of δ_1 and of γ_i for $i \ge 2$) for which both (3.6.7) and (3.6.8) hold (where $\delta_i = \gamma_i$ for $i \ge 3$).

As before, either the result follows or we can assume that (3.6.6) holds and for all places v we have

$$\widehat{h}_v(\delta_2) \le \frac{1}{q^d}\max_{i\ne 2}\widehat{h}_v(\delta_i).$$

From this and (3.6.9) we get

$$\widehat{h}_v(\delta_2) \le \frac{1}{q^d}\max_{1\le i\le r}\widehat{h}_v(\gamma_i)$$

for all places v.

Continuing this process for all i up to r, either we obtain (3.6.2), or we find a subsequence of $\mathbb{N}$, polynomials $R_{n,i}$ satisfying (3.6.6) and $\delta_1, \dots, \delta_r$ (given by Φ-linear combinations of the original γ_i's) satisfying (3.6.7) and (3.6.8), and such that

$$\widehat{h}_v(\delta_j) \le \frac{1}{q^d}\max_{1\le i\le r}\widehat{h}_v(\gamma_i) \tag{3.6.10}$$

for all j and all places v. By summing over $v \in S_0$, we find

$$\widehat{h}(\delta_j) \le \sum_{v\in S_0}\frac{1}{q^d}\max_{1\le i\le r}\widehat{h}_v(\gamma_i) \le \frac{1}{q^d}\sum_{1\le i\le r}\widehat{h}(\gamma_i)$$

for any j.

Repeating the above procedure with the δ_i's in place of the γ_i's, either we can find v such that (3.6.2) holds, or we construct $\epsilon_1, \dots, \epsilon_r$ such that for all j and v we have

$$\widehat{h}_v(\epsilon_j) \le \frac{1}{q^d}\max_{1\le i\le r}\widehat{h}_v(\delta_i).$$

This, combined with (3.6.10), gives

$$\widehat{h}_v(\epsilon_j) \le \frac{1}{q^{2d}} \max_{1\le i\le r} \widehat{h}_v(\gamma_i)$$

for all j and v. Summing over $v \in S_0$ we find

$$\widehat{h}(\epsilon_j) \le \frac{1}{q^{2d}} \sum_{1\le i\le r} \widehat{h}(\gamma_i)$$

for all j. More generally, repeating this process ℓ times, one would find elements $\alpha^{(\ell)} \in \Gamma$ for which

$$\widehat{h}\left(\alpha^{(\ell)}\right) \le \frac{1}{q^{\ell d}} \sum_{1\le i\le r} \widehat{h}(\gamma_i).$$

However, by Lemma 3.4 there exists a positive constant c_0 such that $\widehat{h}(x) \ge c_0$ for all non-torsion $x \in K$. Since Γ is torsion-free, the process has to stop at some finite step (otherwise we would get that $\sum_{i=1}^{r} \Phi_{P_{n,i}}(\gamma_i) = 0$ for large enough n, which is a contradiction); therefore, there exists a place v for which (3.6.2) must hold. □

Now we are ready to prove Theorem 1.1 which follows from Proposition 3.6 coupled with Proposition 3.1 using an identical strategy as the proof of [GT07, Theorem 2.4].

Proof of Theorem 1.1. Let $(\gamma_i)_i$ be a finite set of generators of Γ as a module over $A = \mathbb{F}_q[t]$. At the expense of replacing S with a larger finite set of places of K, we may assume S contains all the places $v \in M_K$ which satisfy at least one of the following properties:

1. $\widehat{h}_v(\gamma_i) > 0$ for some $1 \le i \le r$.
2. $|\gamma_i|_v > 1$ for some $1 \le i \le r$.
3. $|\alpha|_v > 1$.
4. Φ has bad reduction at v.

Expanding the set S leads only to (possible) extension of the set of S-integral points in Γ with respect to α. Clearly, for every $\gamma \in \Gamma$, and for every $v \notin S$ we have $|\gamma|_v \le 1$. Therefore, using 2. and 3., we obtain

(3.6.11)
$$\gamma \in \Gamma \text{ is } S\text{-integral with respect to } \alpha \iff |\gamma - \alpha|_v = 1 \text{ for all } v \in M_K \setminus S.$$

Moreover, using 1. from above, we conclude that for every $\gamma \in \Gamma$, and for every $v \notin S$, we have $\widehat{h}_v(\gamma) = 0$ (see [GT07, Fact 3.13]).

Next we observe that it suffices to prove Theorem 1.1 under the assumption that Γ is a free Φ-submodule. Indeed, because $A = \mathbb{F}_q[t]$ is a principal ideal domain, Γ is a direct sum of its finite torsion submodule Γ_{tor} and a free Φ-submodule Γ_1 of rank r, say. Therefore,

$$\Gamma = \bigcup_{\gamma \in \Gamma_{\text{tor}}} \gamma + \Gamma_1.$$

If we show that for every $\gamma_0 \in \Gamma_{\rm tor}$ there are finitely many $\gamma_1 \in \Gamma_1$ such that γ_1 is S-integral with respect to $\alpha - \gamma_0$, then we obtain the conclusion of Theorem 1.1 for Γ and α (see (3.6.11)).

Thus from now on, we assume Γ is a free Φ-submodule of rank r. Let $\gamma_1, \dots, \gamma_r$ be a basis for Γ as an $\mathbb{F}_q[t]$-module. We reason by contradiction. Let

$$\sum_{i=1}^{r} \Phi_{P_{n,i}}(\gamma_i) \in \Gamma$$

be an infinite sequence of elements S-integral with respect to α. Because of the S-integrality assumption (along with the assumptions on S), we conclude that for every $v \notin S$ and for every n, we have

$$\frac{\log |\sum_{i=1}^r \Phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} = 0.$$

Thus, using the product formula, we see that

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{v\in S} \frac{\log |\sum_{i=1}^r \Phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \\
&= \limsup_{n\to\infty} \sum_{v\in M_K} \frac{\log |\sum_{i=1}^r \Phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \\
&= 0.
\end{aligned}$$

On the other hand, by Proposition 3.6, there is some place $w \in S$ and some number $c_3 > 0$ such that

$$\limsup_{n\to\infty} \frac{\log |\sum_{i=1}^r \Phi_{P_{n,i}}(\gamma_i) - \alpha|_w}{\sum_{i=1}^r q^{d \deg P_{n,i}}} = c_3 > 0.$$

So, using Lemma 3.1, we see that

$$\begin{aligned}
&\limsup_{n\to\infty} \sum_{v\in S} \frac{\log |\sum_{i=1}^r \Phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \\
&\geq \sum_{\substack{v\in S\\ v\neq w}} \liminf_{n\to\infty} \frac{\log |\sum_{i=1}^r \Phi_{P_{n,i}}(\gamma_i) - \alpha|_v}{\sum_{i=1}^r q^{d \deg P_{n,i}}} + \limsup_{n\to\infty} \frac{\log |\sum_{i=1}^r \Phi_{P_{n,i}}(\gamma_i) - \alpha|_w}{\sum_{i=1}^r q^{d \deg P_{n,i}}} \\
&\geq 0 + c_3 \\
&> 0.
\end{aligned}$$

Thus, we have a contradiction which shows that there cannot be infinitely many elements of Γ which are S-integral with respect to α. □

References

[BGT06] J. P. Bell, D. Ghioca and T. J. Tucker, *The Dynamical Mordell-Lang Conjecture*, Mathematical Surveys and Monographs, 2016, vol. **210**, American Mathematical Society, Providence, RI, xiv+280 pp.

[Bos99] V. Bosser, *Minorations de formes linéaires de logarithmes pour les modules de Drinfeld*, J. Number Theory **75** (1999), no. 2, 279–323.

[Bre05] F. Breuer, *The André-Oort conjecture for products of Drinfeld modular curves*, J. reine angew. Math. **579** (2005), 115–144.

[Den92a] L. Denis, *Géométrie diophantienne sur les modules de Drinfel′d*, The arithmetic of function fields (Columbus, OH, 1991), Ohio State Univ. Math. Res. Inst. Publ., vol. 2, de Gruyter, Berlin, 1992, pp. 285–302.

[Den92b] L. Denis, *Hauteurs canoniques et modules de Drinfel′d*, Math. Ann. **294** (1992), no. 2, 213–223.

[EY03] B. Edixhoven and A. Yafaev, *Subvarieties of Shimura type*, Ann. of Math. (2) **157** (2003), no. 2, 621–645.

[Ghi05] D. Ghioca, *The Mordell-Lang theorem for Drinfeld modules*, Int. Math. Res. Not. (2005), no. 53, 3273–3307.

[Ghi06] D. Ghioca, *Equidistribution for torsion points of a Drinfeld module*, Math. Ann. **336** (2006), no. 4, 841–865.

[Ghi07a] D. Ghioca, *The Lehmer inequality and the Mordell-Weil theorem for Drinfeld modules*, J. Number Theory **122** (2007), no. 1, 37–68.

[Ghi07b] D. Ghioca, *The local Lehmer inequality for Drinfeld modules*, J. Number Theory **123** (2007), no. 2, 426–455.

[Gos96] D. Goss, *Basic structures of function field arithmetic*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 35, Springer-Verlag, Berlin, 1996.

[GT07] D. Ghioca and T. J. Tucker, *Siegel's theorem for Drinfeld modules*, Math Ann. **339** (2007), no. 1, 37–60.

[GT08a] D. Ghioca and T. J. Tucker, *Equidistribution and integral points for Drinfeld modules*, Trans. Amer. Math. Soc. **360** (2008), no. 7, 3839–3856.

[GT08b] D. Ghioca and T. J. Tucker, *A dynamical version of the Mordell-Lang conjecture for the additive group*, Compositio Math. **144** (2008), no. 2, 304–316.

[GT09] D. Ghioca and T. J. Tucker, *Periodic points, linarizing maps, and the dynamical Mordell-Lang problem*, J. Number Theory **129** (2009), no. 6, 1392–1403.

[Sca02] T. Scanlon, *Diophantine geometry of the torsion of a Drinfeld module*, J. Number Theory **97** (2002), no. 1, 10–25.

[Ser97] J.-P. Serre, *Lectures on the Mordell-Weil theorem*, third ed., Aspects of Mathematics, Friedr. Vieweg & Sohn, Braunschweig, 1997, Translated from the French and edited by Martin Brown from notes by Michel Waldschmidt, With a foreword by Brown and Serre.

[Sie29] C. L. Siegel, *Über einige anwendungen diophantisher approximationen*, Abh. Preuss. Akad. Wiss. Phys. Math. Kl. (1929), 41–69.

[SUZ97] L. Szpiro, E. Ullmo, and S. Zhang, *Equirépartition des petits points*, Invent. Math. **127** (1997), 337–347.

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